Nonlinear Analysis 71 (2009) 1504–1517

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Variational problems with fractional derivatives: Invariance conditions and Nöther's theorem ☆

Teodor M. Atanacković [a], Sanja Konjik [b,*], Stevan Pilipović [c], Srboljub Simić [a]

[a] Faculty of Technical Sciences, Institute of Mechanics, University of Novi Sad, Trg Dositeja Obradovića 6, 21000 Novi Sad, Serbia

[b] Faculty of Agriculture, Department of Agricultural Engineering, University of Novi Sad, Trg Dositeja Obradovića 8, 21000 Novi Sad, Serbia

[c] Faculty of Science and Mathematics, Department of Mathematics and Informatics, University of Novi Sad, Trg Dositeja Obradovića 4, 21000 Novi Sad, Serbia

ARTICLE INFO

Article history:
Received 23 September 2008
Accepted 14 December 2008

MSC:
primary 49K05
secondary 26A33

PACS:
02.30.Xx
45.10.Hj

Keywords:
Variational problem
Riemann–Liouville fractional derivative
Variational symmetry
Infinitesimal criterion
Nöther's theorem
Conservation laws
Approximations

ABSTRACT

A variational principle for Lagrangian densities containing derivatives of real order is formulated and the invariance of this principle is studied in two characteristic cases. Necessary and sufficient conditions for an infinitesimal transformation group (basic Nöther's identity) are obtained. These conditions extend the classical results, valid for integer order derivatives. A generalization of Nöther's theorem leading to conservation laws for fractional Euler–Lagrangian equation is obtained as well. Results are illustrated by several concrete examples. Finally, an approximation of a fractional Euler–Lagrangian equation by a system of integer order equations is used for the formulation of an approximated invariance condition and corresponding conservation laws.

1. Introduction

There are two distinct approaches in the formulation of fractional differential equations of models in various branches of science, e.g. physics. In the first one, differential equations containing integer order derivatives are modified by replacing one or more of them with fractional ones (derivatives of real order). In the second approach, which we will follow in the sequel, one starts with a variational formulation of a physical process in which one modifies the Lagrangian density by replacing integer order derivatives with fractional ones. Then the action integral in the sense of Hamilton is minimized and the governing equation of a physical process is obtained. Hence, one is faced with the following problem: find minima (or maxima) of a functional

$$\mathcal{L}[u] = \int_A^B L(t, u(t), {}_a\mathrm{D}_t^\alpha u)\,\mathrm{d}t, \quad 0 < \alpha < 1, \tag{1}$$

☆ The work supported by Projects 144016 and 144019 of the Serbian Ministry of Science and START-project Y-237 of the Austrian Science Fund.

* Corresponding author. Fax: +381 21459989.
E-mail addresses: atanackovic@uns.ns.ac.yu (T.M. Atanacković), kinjoki@uns.ns.ac.yu (S. Konjik), pilipovic@im.ns.ac.yu (S. Pilipović), ssimic@uns.ns.ac.yu (S. Simić).

0362-546X/$ – see front matter
doi:10.1016/j.na.2008.12.043

where ${}_aD_t^\alpha u$ is the left Riemann–Liouville fractional derivative, under certain assumptions on a Lagrangian L, as well as on functions u among which minimizers are sought. After this, the Euler–Lagrange equations are formed for the modified Lagrangian, which leads to equations that intrinsically characterize a physical process. This approach has a more sound physical basis (see e.g. [1–5]). It provides the possibility of formulating conservation laws via Nöther's theorem. Conservation laws are very important in particle reaction physics for example, where they are often postulated as conservation principles [6]. The central point is the fact that equations of a model, i.e., Euler–Lagrange equations, give minimizers of a functional. As a rule, fractional differential equations obtained through Euler–Lagrange equations contain left *and* right fractional derivatives, which makes this approach more delicate.

In the last years fractional calculus has become popular as a useful tool for solving problems from various fields [7,8] (see also [9–19]).

The study of fractional variational problems also has a long history. Riewe [20,21] investigated nonconservative Lagrangian and Hamiltonian mechanics and for those cases formulated a version of the Euler–Lagrange equations. O. P. Agrawal continued the study of the fractional Euler–Lagrange equations [22–24], for general fractional variational problems involving Riemann–Liouville, Caputo and Riesz fractional derivatives. Frederico and Torres [25] studied invariance properties of fractional variational problems with Nöther-type theorems by introducing a new concept of fractional conserved quantity which is not constant in time, so the term conserved quantity is not clear. Conservation laws and Hamiltonian-type equations for the fractional action principles have also been derived in [26]. There are nowadays numerous applications of fractional variational calculus, see e.g. [27,28,1–3,29–32,16,33–35].

We can summarize the novelties of our paper as follows. First, we derived infinitesimal criterion for a local one-parameter group of transformations to be a variational symmetry group for the fractional variational problem (1). In previous work [25, 36,37] this was done only in a special case. Moreover, we separately consider cases when the lower bound a in the left Riemann–Liouville fractional derivative is not transformed (Section 2.1), and when a is transformed (Section 2.2). Both cases have their physical interpretation. In the case when fractional derivatives model memory effects the lower bound a in the definition of derivative should not be varied. However, if fractional derivatives model nonlocal interactions (e.g. in nonlocal elasticity) the lower bound has to be varied. Second, we give a Nöther-type theorem in terms of conservation laws as it is done in the classical theory. This approach preserves the essential property of conserved quantities to be constant in time. Third novelty is the approximation procedure which is given for the Euler–Lagrange equations and infinitesimal criterion, therefore also for Nöther's theorem. By the use of additional assumptions we obtain that appropriate sequences of classical Euler–Lagrange equations, infinitesimal criteria and conservation laws converge in the sense of an appropriate space of generalized functions to the corresponding Euler–Lagrange equations, infinitesimal criterion and conservation laws of the fractional variational problem (1). There are also a number of illustrative examples which complete the theory.

The paper is organized as follows. To the end of this introductory section we provide basic notions and definitions from the calculus with derivatives and integrals of any real order, and recall the Euler–Lagrange equations of (1) for $(A, B) \subseteq (a, b)$ (so far only the case $(A, B) = (a, b)$ was treated, cf. [22,23]). Section 2 is devoted to the study of variational symmetry groups and invariance conditions. In Section 3 we derive a version of Nöther's theorem which establishes a relation between fractional variational symmetries and conservation laws for the Euler–Lagrange equation, which have the important property of being constant in time. We use in Section 4 the approximation of the Riemann–Liouville fractional derivative by a finite sum of classical derivatives. In this way and by introducing a suitable space of analytic functions we obtain approximated Euler–Lagrange equations, approximated local Lie group actions with corresponding infinitesimal criteria and Nöther theorems.

1.1. Preliminaries

In the sequel we briefly recall some basic facts from the fractional calculus.

For $u \in L^1([a, b])$, $\alpha > 0$ and $t \in [a, b]$ we define the left, resp. right Riemann–Liouville fractional integral of order α, as

$$ {}_aI_t^\alpha u = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\theta)^{\alpha-1} u(\theta) \mathrm{d}\theta, \qquad \text{resp. } {}_tI_b^\alpha u = \frac{1}{\Gamma(\alpha)} \int_t^b (\theta - t)^{\alpha-1} u(\theta) \mathrm{d}\theta. $$

Left, resp. right Riemann–Liouville fractional derivative of order α, $0 \le \alpha < 1$, is well defined for an absolutely continuous function u in $[a, b]$, i.e. $u \in AC([a, b])$, and $t \in [a, b]$ as

$$ {}_aD_t^\alpha u = \frac{\mathrm{d}}{\mathrm{d}t} {}_aI_t^{1-\alpha} u = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t \frac{u(\theta)}{(t-\theta)^\alpha} \mathrm{d}\theta, \tag{2} $$

resp.

$$ {}_tD_b^\alpha u = \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right) {}_tI_b^{1-\alpha} u = \frac{1}{\Gamma(1-\alpha)} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right) \int_t^b \frac{u(\theta)}{(\theta-t)^\alpha} \mathrm{d}\theta. $$

If $f, g \in AC([a, b])$ and $0 \le \alpha < 1$ the following fractional integration by parts formula holds:

$$ \int_a^b f(t)\, {}_aD_t^\alpha g \mathrm{d}t = \int_a^b g(t)\, {}_tD_b^\alpha f \mathrm{d}t. \tag{3} $$

For the left Riemann–Liouville fractional derivative (2) of order α $(0 \le \alpha < 1)$ we have:

$$ {}_a\mathrm{D}_t^\alpha u = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{\dot{u}(\theta)}{(t-\theta)^\alpha} \mathrm{d}\theta + \frac{1}{\Gamma(1-\alpha)} \frac{u(a)}{(t-a)^\alpha}, \quad t \in [a,b]. \tag{4} $$

The integral on the right hand side is called the left Caputo fractional derivative of order α and is denoted by ${}_a^c\mathrm{D}_t^\alpha u$. Similarly, the right Caputo fractional derivative ${}_t^c D_b^\alpha u$ is defined.

It follows from (4) that the left Riemann–Liouville equals to the left Caputo fractional derivative in the case $u(a) = 0$ (the analogue holds for the right derivatives under the assumption $u(b) = 0$). The same condition, i.e. $u(a) = 0$, provides that $\frac{\mathrm{d}}{\mathrm{d}t}\, {}_a\mathrm{D}_t^\alpha u = {}_a\mathrm{D}_t^\alpha \frac{\mathrm{d}}{\mathrm{d}t} u$.

We introduce the notation which will be used throughout this paper: derivatives of Lagrangian $L = L(t, u(t), {}_a\mathrm{D}_t^\alpha u)$ with respect to the first, second and third variable will be denoted by $\frac{\partial L}{\partial t}$, $\frac{\partial L}{\partial u}$ and $\frac{\partial L}{\partial\, {}_a\mathrm{D}_t^\alpha u}$, or by $\partial_1 L$, $\partial_2 L$ and $\partial_3 L$, respectively.

1.2. Euler–Lagrange equations

As stated in Introduction, we solve a fractional variational problem

$$ \mathcal{L}[u] = \int_A^B L(t, u(t), {}_a\mathrm{D}_t^\alpha u)\mathrm{d}t \to \min, \quad 0 < \alpha < 1. \tag{5} $$

(A, B) is a subinterval of (a, b), and Lagrangian L is a function in $(a, b) \times \mathbb{R} \times \mathbb{R}$ such that

$$ \left.\begin{array}{l} L \in \mathcal{C}^1((a,b)\times\mathbb{R}\times\mathbb{R}) \\ \text{and} \\ t \mapsto \partial_2 L(t, u(t), {}_a\mathrm{D}_t^\alpha u) \text{ is integrable in } (a,b) \text{ and} \\ t \mapsto \partial_3 L(t, u(t), {}_a\mathrm{D}_t^\alpha u) \in AC([a,b]), \text{ for every } u \in AC([a,b]) \end{array}\right\}. \tag{6} $$

Solutions to (5) are sought among all absolutely continuous functions in $[a, b]$, which in addition satisfy condition $u(a) = a_0$, for a fixed $a_0 \in \mathbb{R}$.

One can consider more general problems with Lagrangians depending also on the right Riemann–Liouville fractional derivative. Our results can be easily formulated in that case, and because of simplicity we shall consider only Lagrangians of the form (5). Moreover, in (5) one can consider the case $\alpha > 1$, but this also does not give any essential novelty and because of that it is skipped.

We will recall results about the Euler–Lagrange equations obtained in [22,23,38]. As we said, we consider the fractional variational problem defined by (5).

Fractional Euler–Lagrange equations, which provide a necessary condition for extremals of a fractional variational problem, have been recently studied in [22,23,38]. Let $A = a$ and $B = b$. Euler–Lagrange equations are obtained in [22,23]:

$$ \frac{\partial L}{\partial u} + {}_t D_b^\alpha \left(\frac{\partial L}{\partial_a D_t^\alpha u} \right) = 0. \tag{7} $$

In terms of the Caputo fractional derivative, the Euler–Lagrange equation (7) is given in [38] and reads:

$$ \frac{\partial L}{\partial u} + {}_t^c D_b^\alpha \left(\frac{\partial L}{\partial_a D_t^\alpha u} \right) + \left.\frac{\partial L}{\partial\, {}_a\mathrm{D}_t^\alpha u}\right|_{t=b} \frac{1}{\Gamma(1-\alpha)} \frac{1}{(b-t)^\alpha} = 0. \tag{8} $$

Euler–Lagrange equations for (5) are derived in [38]:

$$ \frac{\partial L}{\partial u} + {}_t^c D_B^\alpha \left(\frac{\partial L}{\partial_a D_t^\alpha u} \right) + \left.\frac{\partial L}{\partial\, {}_a\mathrm{D}_t^\alpha u}\right|_{t=B} \frac{1}{\Gamma(1-\alpha)} \frac{1}{(B-t)^\alpha} = 0, \quad t \in (A, B) \tag{9} $$

$$ {}_t D_B^\alpha \left(\frac{\partial L}{\partial_a D_t^\alpha u} \right) - {}_t D_A^\alpha \left(\frac{\partial L}{\partial_a D_t^\alpha u} \right) = 0, \quad t \in (a, A). \tag{10} $$

Eq. (9) is equivalent to

$$ \frac{\partial L}{\partial u} + {}_t D_B^\alpha \left(\frac{\partial L}{\partial_a D_t^\alpha u} \right) = 0, \quad t \in (A, B). \tag{11} $$

Remark 1. If we replace the Lagrangian in (5) by $L(t, u(t), {}_a\mathrm{D}_t^\alpha u, \dot{u}(t))$ then Euler–Lagrange equations will be also equipped with a 'classical' term $-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{u}}$ on the left hand side. It can be shown that in this case infinitesimal criterion and conservation law can be obtained by combining the results of the present analysis (see Theorems 5, 11 and 15) and the classical theory (see e.g. [39]).

2. Infinitesimal invariance

Let G be a local one-parameter group of transformations acting on a space of independent and dependent variables as follows: $(\bar{t},\bar{u}) = g_\eta \cdot (t,u) = (\Xi_\eta(t,u), \Psi_\eta(t,u))$, for smooth functions Ξ_η and Ψ_η, and $g_\eta \in G$. Let

$$\mathbf{v} = \tau(t,u)\frac{\partial}{\partial t} + \xi(t,u)\frac{\partial}{\partial u}$$

be the infinitesimal generator of G. Then we also have

$$\begin{aligned}\bar{t} &= t + \eta\tau(t,u) + o(\eta)\\ \bar{u} &= u + \eta\xi(t,u) + o(\eta).\end{aligned} \tag{12}$$

We introduce the following notation (cf., e.g., [40]): $\Delta t = \frac{\mathrm{d}}{\mathrm{d}\eta}|_{\eta=0}(\bar{t}-t)$ and $\Delta u = \frac{\mathrm{d}}{\mathrm{d}\eta}|_{\eta=0}(\bar{u}(\bar{t}) - u(t))$. More precisely, the notations Δt and Δu denote $\lim_{\eta\to 0}\frac{\bar{t}(\eta)-t}{\eta}$ and $\lim_{\eta\to 0}\frac{\bar{u}(\bar{t},\eta)-u(t)}{\eta}$, respectively. It follows from (12) that $\Delta t = \tau$, and writing the Taylor expansion of $\bar{u}(\bar{t}) = \bar{u}(t+\eta\tau(t,u)+o(\eta))$ at $\eta = 0$ yields that $\Delta u = \xi$.

On the other hand, if we write Taylor expansion of $\bar{u}(\bar{t})$ at $\bar{t} = t$ we obtain

$$\Delta u = \left.\frac{\mathrm{d}}{\mathrm{d}\eta}\right|_{\eta=0}(\bar{u}(t) - u(t)) + \dot{u}\Delta t,$$

or, if we introduce the Lagrangian variation

$$\delta u := \left.\frac{\mathrm{d}}{\mathrm{d}\eta}\right|_{\eta=0}(\bar{u}(t) - u(t)) \tag{13}$$

we obtain

$$\Delta u = \delta u + \dot{u}\Delta t.$$

Thus, we have

$$\delta u = \xi - \tau\dot{u}.$$

Note that $\delta t = 0$.

In the same way we can define ΔF and δF of an arbitrary absolutely continuous function $F = F(t, u(t), \dot{u}(t))$:

$$\begin{aligned}\Delta F &= \left.\frac{\mathrm{d}}{\mathrm{d}\eta}\right|_{\eta=0}\big(F(\bar{t},\bar{u}(\bar{t}),\dot{\bar{u}}(\bar{t})) - F(t,u(t),\dot{u}(t))\big) = \frac{\partial F}{\partial t}\Delta t + \frac{\partial F}{\partial u}\Delta u + \frac{\partial F}{\partial \dot{u}}\Delta\dot{u}\\ \delta F &= \left.\frac{\mathrm{d}}{\mathrm{d}\eta}\right|_{\eta=0}\big(F(t,\bar{u}(t),\dot{\bar{u}}(t)) - F(t,u(t),\dot{u}(t))\big) = \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial \dot{u}}\delta\dot{u},\end{aligned}$$

and

$$\Delta F = \delta F + \dot{F}\Delta t.$$

It will be apparent in the forthcoming sections that, if we want to find an infinitesimal criterion, we need to know $\Delta\, {}_a\mathrm{D}_t^\alpha u$ and $\Delta\mathcal{L}$. Therefore, we have to transform the left Riemann–Liouville fractional derivative of u under the action of a local one-parameter group of transformations (12). There are two different cases which will be considered separately. In the first one, the lower bound a in ${}_a\mathrm{D}_t^\alpha u$ is not transformed, while in the second case a is transformed in the same way as the independent variable t. Physically, the first case is important when ${}_a\mathrm{D}_t^\alpha u$ represents memory effects, and the second one is important when action on a distance is involved.

2.1. The case when a in ${}_a\mathrm{D}_t^\alpha u$ is not transformed

In this section we consider a local group of transformations G which transforms $t \in (A,B)$ into $\bar{t} \in (\bar{A},\bar{B})$ so that both intervals remain subintervals of (a,b), but the action of G has no effect on the lower bound a in ${}_a\mathrm{D}_t^\alpha u$, i.e., $\tau(a,u(a)) = 0$. So, suppose that G acts on t, u and ${}_a\mathrm{D}_t^\alpha u$ in the following way:

$$g_\eta \cdot (t, u, {}_a\mathrm{D}_t^\alpha u) := (\bar{t}, \bar{u}, {}_a D_{\bar{t}}^\alpha \bar{u}),$$

where $\bar{t}$ and $\bar{u}$ are defined by (12). In this case we have:

Lemma 2. *Let $u \in AC([a,b])$ and let G be a local one-parameter group of transformations given by (12). Then*

$$\Delta\, {}_a\mathrm{D}_t^{\alpha}\, u = {}_a\mathrm{D}_t^{\alpha}\, \delta u + \frac{\mathrm{d}}{\mathrm{d}t}\, {}_a\mathrm{D}_t^{\alpha}\, u \cdot \tau(t, u(t)),$$

where

$$\delta\, {}_a\mathrm{D}_t^{\alpha}\, u = \left.\frac{\mathrm{d}}{\mathrm{d}\eta}\right|_{\eta=0} \left({}_aD_{\bar{t}}^{\alpha}\bar{u} - {}_a\mathrm{D}_t^{\alpha}\, u\right) = {}_a\mathrm{D}_t^{\alpha}\, \delta u.$$

Proof. To prove that $\delta\, {}_a\mathrm{D}_t^{\alpha}\, u = {}_a\mathrm{D}_t^{\alpha}\, \delta u$ it is enough to apply the definition of the Lagrangian variation (13). Also by definition we have

$$\Delta\, {}_a\mathrm{D}_t^{\alpha}\, u = \left.\frac{\mathrm{d}}{\mathrm{d}\eta}\right|_{\eta=0} \left({}_aD_{\bar{t}}^{\alpha}\bar{u} - {}_a\mathrm{D}_t^{\alpha}\, u\right).$$

Thus,

$$\begin{aligned}
\Delta\, {}_a\mathrm{D}_t^{\alpha}\, u &= \frac{1}{\Gamma(1-\alpha)} \left.\frac{\mathrm{d}}{\mathrm{d}\eta}\right|_{\eta=0} \left[\frac{\mathrm{d}}{\mathrm{d}\bar{t}} \int_a^{\bar{t}} \frac{\bar{u}(\theta)}{(\bar{t}-\theta)^{\alpha}} \mathrm{d}\theta \pm \frac{\mathrm{d}}{\mathrm{d}t} \int_a^{t} \frac{\bar{u}(\theta)}{(t-\theta)^{\alpha}}\, \mathrm{d}\theta - \frac{\mathrm{d}}{\mathrm{d}t} \int_a^{t} \frac{u(\theta)}{(t-\theta)^{\alpha}} \mathrm{d}\theta \right] \\
&= {}_aD_t^{\alpha}\delta u + \left.\frac{\mathrm{d}}{\mathrm{d}\eta}\right|_{\eta=0} \left({}_aD_{\bar{t}}^{\alpha}\bar{u} - {}_a\mathrm{D}_t^{\alpha}\, \bar{u}\right) \\
&= {}_aD_t^{\alpha}\delta u + \frac{\mathrm{d}}{\mathrm{d}\bar{t}} \left.\frac{\mathrm{d}\bar{t}}{d\eta}\right|_{\eta=0} \left({}_aD_{\bar{t}}^{\alpha}\bar{u} - {}_a\mathrm{D}_t^{\alpha}\, \bar{u}\right) \\
&= {}_aD_t^{\alpha}\delta u + \frac{\mathrm{d}}{\mathrm{d}t}\, {}_a\mathrm{D}_t^{\alpha}\, u \cdot \tau(t, u(t)). \quad \square
\end{aligned}$$

Lemma 3. *Let $\mathcal{L}[u]$ be a functional of the form $\mathcal{L}[u] = \int_A^B L(t, u(t), {}_aD_t^{\alpha}u)\mathrm{d}t$, where u is an absolutely continuous function in $[a,b]$, $(A,B) \subseteq (a,b)$ and L satisfies (6). Let G be a local one-parameter group of transformations given by (12). Then*

$$\Delta\mathcal{L} = \delta\mathcal{L} + (L\Delta t)\Big|_A^B,$$

where $\delta\mathcal{L} = \int_a^b \delta L \mathrm{d}t$.

Proof. Again it is clear that $\delta\mathcal{L} = \int_a^b \delta L\, \mathrm{d}t$. For $\Delta\mathcal{L}$ we have

$$\begin{aligned}
\Delta\mathcal{L} &= \left.\frac{\mathrm{d}}{\mathrm{d}\eta}\right|_{\eta=0} \left(\int_{\bar{A}}^{\bar{B}} L(\bar{t}, \bar{u}(\bar{t}), {}_aD_{\bar{t}}^{\alpha}\bar{u})\mathrm{d}\bar{t} - \int_A^B L(t, u(t), {}_aD_t^{\alpha}u)\mathrm{d}t \right) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}\eta}\right|_{\eta=0} \left(\int_A^B L(\bar{t}, \bar{u}(\bar{t}), {}_aD_{\bar{t}}^{\alpha}\bar{u})\, (1 + \eta\dot{\tau}(t, u(t)))\mathrm{d}t - \int_A^B L(t, u(t), {}_aD_t^{\alpha}u)\mathrm{d}t \right) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}\eta}\right|_{\eta=0} \left(\int_A^B L(\bar{t}, \bar{u}(\bar{t}), {}_aD_{\bar{t}}^{\alpha}\bar{u})\mathrm{d}t - \int_A^B L(t, u(t), {}_aD_t^{\alpha}u)\mathrm{d}t \right) + \int_A^B L(t, u(t), {}_aD_t^{\alpha}u)\dot{\tau}(t, u(t))\mathrm{d}t.
\end{aligned}$$

This further yields

$$\begin{aligned}
\Delta\mathcal{L} &= \int_A^B \Delta L(t, u(t), {}_aD_t^{\alpha}u)\mathrm{d}t + \int_A^B L(t, u(t), {}_aD_t^{\alpha}u)\dot{\tau}(t, u(t))\mathrm{d}t \\
&= \int_A^B \delta L(t, u(t), {}_aD_t^{\alpha}u)\mathrm{d}t + \int_A^B \frac{\mathrm{d}}{\mathrm{d}t} L(t, u(t), {}_aD_t^{\alpha}u)\, \tau(t, u(t))\mathrm{d}t + \int_A^B L(t, u(t), {}_aD_t^{\alpha}u)\, \dot{\tau}(t, u(t))\mathrm{d}t \\
&= \int_A^B \delta L \mathrm{d}t + (L\tau)\Big|_A^B
\end{aligned}$$

and the claim is proved. $\square$

We define a variational symmetry group of the fractional variational problem (5), which we call a fractional variational symmetry group:

Definition 4. A local one-parameter group of transformations G (12) is a variational symmetry group of the fractional variational problem (5) if the following conditions holds: for every $[A', B'] \subset (A, B)$, $u = u(t) \in AC([A', B'])$ and $g_\eta \in G$ such that $\bar{u}(\bar{t}) = g_\eta \cdot u(\bar{t})$ is in $AC([\bar{A}', \bar{B}'])$, we have

$$\int_{\bar{A}'}^{\bar{B}'} L(\bar{t}, \bar{u}(\bar{t}), {}_a D_{\bar{t}}^\alpha \bar{u}) \mathrm{d}\bar{t} = \int_{A'}^{B'} L(t, u(t), {}_a \mathrm{D}_t^\alpha u) \mathrm{d}t. \tag{14}$$

We are now able to prove the following infinitesimal criterion:

Theorem 5. *Let $\mathcal{L}[u]$ be a fractional variational problem (5) and let G be a local one-parameter transformation group (12) with the infinitesimal generator $\mathbf{v} = \tau(t, u)\partial_t + \xi(t, u)\partial_u$. Then G is a variational symmetry group of $\mathcal{L}$ if and only if*

$$\tau \frac{\partial L}{\partial t} + \xi \frac{\partial L}{\partial u} + \left({}_a\mathrm{D}_t^\alpha (\xi - \dot{u}\tau) + \left(\frac{\mathrm{d}}{\mathrm{d}t} {}_a\mathrm{D}_t^\alpha u \right) \tau \right) \frac{\partial L}{\partial\, {}_a\mathrm{D}_t^\alpha u} + L\dot{\tau} = 0. \tag{15}$$

Proof. Suppose that G is a variational symmetry group of $\mathcal{L}$. Then (14) holds for all subintervals (A', B') of (A, B) with closure $[A', B'] \subset (A, B)$. We have:

$$\begin{aligned}
\Delta\mathcal{L} &= \int_{A'}^{B'} \delta L \mathrm{d}t + (L\Delta t)\Big|_{A'}^{B'} \\
&= \int_{A'}^{B'} \left(\frac{\partial L}{\partial u}\delta u + \frac{\partial L}{\partial\, {}_a\mathrm{D}_t^\alpha u} \delta({}_a\mathrm{D}_t^\alpha u) \right) \mathrm{d}t + (L\Delta t)\Big|_{A'}^{B'} \\
&= \int_{A'}^{B'} \left(\frac{\partial L}{\partial u}(\Delta u - \dot{u}\Delta t) + \frac{\partial L}{\partial\, {}_a\mathrm{D}_t^\alpha u} {}_a\mathrm{D}_t^\alpha (\Delta u - \dot{u}\Delta t) \right) \mathrm{d}t + (L\Delta t)\Big|_{A'}^{B'} \\
&= \int_{A'}^{B'} \left(\frac{\partial L}{\partial u}\Delta u + \frac{\partial L}{\partial\, {}_a\mathrm{D}_t^\alpha u} {}_a\mathrm{D}_t^\alpha \Delta u \right) \mathrm{d}t - \int_{A'}^{B'} \left(\frac{\partial L}{\partial u}(\dot{u}\Delta t) + \frac{\partial L}{\partial\, {}_a\mathrm{D}_t^\alpha u} {}_a\mathrm{D}_t^\alpha (\dot{u}\Delta t) \pm \frac{\partial L}{\partial t}\Delta t \right) \mathrm{d}t + (L\Delta t)\Big|_{A'}^{B'} \\
&= *.
\end{aligned}$$

Applying the Leibniz rule for $\frac{\mathrm{d}}{\mathrm{d}t}(L\Delta t)$ we replace $\frac{\partial L}{\partial t}\Delta t + \frac{\partial L}{\partial u}\dot{u}\Delta t$ by $\frac{\mathrm{d}}{\mathrm{d}t}(L\Delta t) - \frac{\partial L}{\partial\, {}_a\mathrm{D}_t^\alpha u}(\frac{\mathrm{d}}{\mathrm{d}t}\, {}_a\mathrm{D}_t^\alpha u)\Delta t - L\Delta t^{(1)}$. Then

$$* = \int_{A'}^{B'} \left(\frac{\partial L}{\partial t}\Delta t + \frac{\partial L}{\partial u}\Delta u + \frac{\partial L}{\partial\, {}_a\mathrm{D}_t^\alpha u} {}_a\mathrm{D}_t^\alpha \Delta u + \frac{\partial L}{\partial\, {}_a\mathrm{D}_t^\alpha u} \left(\left(\frac{\mathrm{d}}{\mathrm{d}t} {}_a\mathrm{D}_t^\alpha u \right) \Delta t - {}_a\mathrm{D}_t^\alpha (\dot{u}\Delta t) \right) + L\Delta t^{(1)} \right) \mathrm{d}t.$$

Since $\Delta\mathcal{L}$ has to be zero in all (A', B') with $[A', B'] \subset (A, B)$, the above integrand has also to be equal zero, i.e.,

$$\tau \frac{\partial L}{\partial t} + \xi \frac{\partial L}{\partial u} + \left({}_a\mathrm{D}_t^\alpha (\xi - \dot{u}\tau) + \left(\frac{\mathrm{d}}{\mathrm{d}t} {}_a\mathrm{D}_t^\alpha u \right) \tau \right) \frac{\partial L}{\partial\, {}_a\mathrm{D}_t^\alpha u} + L\dot{\tau} = 0.$$

where we have used that $\Delta u = \xi$ and $\Delta t = \tau$. Hence, necessity of the statement is proved. To prove that condition (15) is also sufficient, we first realize that if (15) holds on every $[A', B'] \subset (A, B)$, then $\Delta\mathcal{L} = 0$ in every $[A', B'] \subset (A, B)$. Thus, integrating $\Delta\mathcal{L}$ from 0 to η we obtain (14), for η near the identity. The proof is now complete. □

In the following example we calculate the transformation of the fractional derivative ${}_a\mathrm{D}_t^\alpha u$ under the group of translations. In Section 2.2 we shall perform such calculation also in the case when a in ${}_a\mathrm{D}_t^\alpha u$ is transformed.

Example 6. Let G be a local one-parameter translation group: $(\bar{t}, \bar{u}) = (t + \eta, u + \eta)$, with the infinitesimal generator $\mathbf{v} = \partial_t + \partial_u$. Then $\bar{u}(\bar{t}) = u(\bar{t} - \eta) + \eta$ and by a straightforward calculation it can be shown that

$${}_aD_{\bar{t}}^\alpha \bar{u} = {}_a\mathrm{D}_t^\alpha (u + \eta) + \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_{a-\eta}^{a} \frac{u(s) + \eta}{(t-s)^\alpha} \mathrm{d}s.$$

2.2. The case when a in ${}_a\mathrm{D}_t^\alpha u$ is transformed

Now let the action of a local one-parameter group of transformations (12) be of the form

$$g_\eta \cdot (t, u, {}_a\mathrm{D}_t^\alpha u) := (\bar{t}, \bar{u}, {}_{\bar{a}} D_{\bar{t}}^\alpha \bar{u}).$$

Thus, the one-parameter group also acts on a and transforms it to $\bar{a}$, where $\bar{a} = g_\eta \cdot t|_{t=a}$. This will also influence the calculation of $\Delta\, {}_a\mathrm{D}_t^\alpha u$ and $\Delta\mathcal{L}$:

Lemma 7. *Let $u \in AC([a,b])$ and let G be a local one-parameter group of transformations (12). Then*

$$\Delta\, {}_a\mathrm{D}_t^\alpha u = {}_a\mathrm{D}_t^\alpha \delta u + \frac{\mathrm{d}}{\mathrm{d}t}\, {}_a\mathrm{D}_t^\alpha u \cdot \tau(t,u(t)) + \frac{\alpha}{\Gamma(1-\alpha)} \frac{u(a)}{(t-a)^{\alpha+1}} \tau(a,u(a)), \tag{16}$$

where

$$\delta\, {}_a\mathrm{D}_t^\alpha u = \left.\frac{\mathrm{d}}{\mathrm{d}\eta}\right|_{\eta=0} \left({}_aD_t^\alpha \bar{u} - {}_a\mathrm{D}_t^\alpha u\right) = {}_a\mathrm{D}_t^\alpha \delta u.$$

Proof. Again using (13) we check that $\delta\, {}_a\mathrm{D}_t^\alpha u = {}_a\mathrm{D}_t^\alpha \delta u$. To prove (16) we start with the definition of $\Delta\, {}_a\mathrm{D}_t^\alpha u$:

$$\Delta\, {}_a\mathrm{D}_t^\alpha u = \left.\frac{\mathrm{d}}{\mathrm{d}\eta}\right|_{\eta=0} \left({}_{\bar{a}}D_{\bar{t}}^\alpha \bar{u} - {}_a\mathrm{D}_t^\alpha u\right).$$

Thus,

$$\begin{aligned}
\Delta\, {}_a\mathrm{D}_t^\alpha u &= \frac{1}{\Gamma(1-\alpha)} \left.\frac{\mathrm{d}}{\mathrm{d}\eta}\right|_{\eta=0} \left[\frac{\mathrm{d}}{\mathrm{d}\bar{t}} \int_{\bar{a}}^{\bar{t}} \frac{\bar{u}(\theta)}{(\bar{t}-\theta)^\alpha}\mathrm{d}\theta \pm \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t \frac{\bar{u}(\theta)}{(t-\theta)^\alpha}\mathrm{d}\theta - \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t \frac{u(\theta)}{(t-\theta)^\alpha}\mathrm{d}\theta\right] \\
&= {}_aD_t^\alpha \delta u + \frac{1}{\Gamma(1-\alpha)} \left.\frac{\mathrm{d}}{\mathrm{d}\eta}\right|_{\eta=0} \left[\frac{\mathrm{d}}{\mathrm{d}\bar{t}} \int_{\bar{a}}^{a} \frac{\bar{u}(\theta)}{(\bar{t}-\theta)^\alpha}\mathrm{d}\theta + \frac{\mathrm{d}}{\mathrm{d}\bar{t}} \int_a^{\bar{t}} \frac{\bar{u}(\theta)}{(\bar{t}-\theta)^\alpha}\mathrm{d}\theta - \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t \frac{\bar{u}(\theta)}{(t-\theta)^\alpha}\mathrm{d}\theta\right] \\
&= {}_aD_t^\alpha \delta u + \frac{\mathrm{d}}{\mathrm{d}t}\, {}_a\mathrm{D}_t^\alpha u \cdot \tau(t,u(t)) + \frac{1}{\Gamma(1-\alpha)} \left.\frac{\mathrm{d}}{\mathrm{d}\eta}\right|_{\eta=0} \frac{\mathrm{d}}{\mathrm{d}\bar{t}} \int_{\bar{a}}^{a} \frac{\bar{u}(\theta)}{(\bar{t}-\theta)^\alpha}\mathrm{d}\theta.
\end{aligned}$$

Note that in the last term we can interchange the order of integration and differentiation with respect to $\bar{t}$. This gives

$$\frac{-\alpha}{\Gamma(1-\alpha)} \left.\frac{\mathrm{d}}{\mathrm{d}\eta}\right|_{\eta=0} \int_{\bar{a}}^{a} \frac{\bar{u}(\theta)}{(\bar{t}-\theta)^\alpha}\mathrm{d}\theta.$$

If we differentiate this integral with respect to η at $\eta = 0$ we eventually obtain (16). □

Lemma 8. *Let $\mathcal{L}[u]$ be a functional of the form $\mathcal{L}[u] = \int_A^B L(t,u(t),{}_aD_t^\alpha u)\mathrm{d}t$, where u is an absolutely continuous function in (a,b), $(A,B) \subseteq (a,b)$ and L satisfies (6). Let G be a local one-parameter group of transformations given by (12). Then*

$$\Delta\mathcal{L} = \delta\mathcal{L} + (L\Delta t)\big|_A^B + \frac{\alpha}{\Gamma(1-\alpha)} \frac{u(a)}{(t-a)^{\alpha+1}} \tau(a,u(a)) \int_A^B \frac{\partial L}{\partial_a D_t^\alpha u}\mathrm{d}t, \tag{17}$$

where $\delta\mathcal{L} = \int_A^B \delta L\mathrm{d}t$.

Proof. Clearly, $\delta\mathcal{L} = \int_a^b \delta L\mathrm{d}t$. On the other hand we have

$$\begin{aligned}
\Delta\mathcal{L} &= \left.\frac{\mathrm{d}}{\mathrm{d}\eta}\right|_{\eta=0} \left(\int_{\bar{A}}^{\bar{B}} L(\bar{t},\bar{u}(\bar{t}),{}_{\bar{a}}D_{\bar{t}}^\alpha \bar{u})\mathrm{d}\bar{t} - \int_A^B L(t,u(t),{}_aD_t^\alpha u)\mathrm{d}t\right) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}\eta}\right|_{\eta=0} \left(\int_A^B L(\bar{t},\bar{u}(\bar{t}),{}_{\bar{a}}D_{\bar{t}}^\alpha \bar{u})\,(1+\eta\dot{\tau}(t,u(t)))\mathrm{d}t - \int_A^B L(t,u(t),{}_aD_t^\alpha u)\mathrm{d}t\right) \\
&= \int_A^B \Delta L\mathrm{d}t + \int_A^B L(t,u(t),{}_aD_t^\alpha u)\,\dot{\tau}(t,u(t))\mathrm{d}t \\
&= \int_A^B \left(\frac{\partial L}{\partial t}\Delta t + \frac{\partial L}{\partial u}\Delta u + \frac{\partial L}{\partial_a D_t^\alpha u}\Delta_a D_t^\alpha u\right)\mathrm{d}t + \int_A^B L(t,u(t),{}_aD_t^\alpha u)\,\dot{\tau}(t,u(t))\mathrm{d}t.
\end{aligned}$$

We now apply (16) to obtain

$$\begin{aligned}
\Delta\mathcal{L} &= \int_A^B \left(\frac{\partial L}{\partial t}\Delta t + \frac{\partial L}{\partial u}(\delta u + \dot{u}\Delta t) + \frac{\partial L}{\partial_a D_t^\alpha u}\left({}_a\mathrm{D}_t^\alpha \delta u + \frac{\mathrm{d}}{\mathrm{d}t}\, {}_a\mathrm{D}_t^\alpha u \cdot \tau(t,u(t))\right.\right. \\
&\quad \left.\left. + \frac{\alpha}{\Gamma(1-\alpha)} \frac{u(a)}{(t-a)^{\alpha+1}} \tau(a,u(a))\right)\right)\mathrm{d}t + \int_A^B L(t,u(t),{}_aD_t^\alpha u)\,\dot{\tau}(t,u(t))\mathrm{d}t \\
&= \int_A^B \delta L\mathrm{d}t + (L\Delta t)\Big|_A^B + \frac{\alpha}{\Gamma(1-\alpha)} \frac{u(a)}{(t-a)^{\alpha+1}} \tau(a,u(a)) \int_A^B \frac{\partial L}{\partial_a D_t^\alpha u}\,\mathrm{d}t,
\end{aligned}$$

$$= \delta\mathcal{L} + (L\Delta t)\big|_A^B + \frac{\alpha}{\Gamma(1-\alpha)} \frac{u(a)}{(t-a)^{\alpha+1}} \tau(a, u(a)) \int_A^B \frac{\partial L}{\partial_a D_t^\alpha u} dt. \quad \square$$

Remark 9. It should be emphasized that the last summand on the right-hand side of (16), as well as of (17), appears only as a consequence of the fact that the action of a transformation group affects also a, on which the left Riemann–Liouville fractional derivative depends. In the case when $u(a) = 0$ or $\alpha = 1$ that term is equal to zero (since $\lim_{\alpha\to 1^-} \Gamma(1-\alpha) = \infty$) and $\Delta\mathcal{L} = \delta\mathcal{L} + (L\Delta t)|_A^B$.

Now we define a variational symmetry group of the fractional variational problem (5) as follows:

Definition 10. A local one-parameter group of transformations G (12) is a variational symmetry group of the fractional variational problem (5) if the following conditions holds: for every $[A', B'] \subset (A, B)$, $u = u(t) \in AC([A', B'])$ and $g_\eta \in G$ such that $\bar{u}(\bar{t}) = g_\eta \cdot u(\bar{t})$ is in $AC([\bar{A}', \bar{B}'])$, we have

$$\int_{\bar{A}'}^{\bar{B}'} L(\bar{t}, \bar{u}(\bar{t}), {}_{\bar{a}}D_{\bar{t}}^\alpha \bar{u}) d\bar{t} = \int_{A'}^{B'} L(t, u(t), {}_aD_t^\alpha u) dt.$$

Recall that we are solving the fractional variational problem (5) among all absolutely continuous functions in $[a, b]$, which additionally satisfy $u(a) = 0$. Therefore, the last term in both (16) and (17) vanishes, and the infinitesimal criterion reads the same as in Theorem 5:

Theorem 11. *Let $\mathcal{L}[u]$ be a fractional variational problem defined by (5) and let G be a local one-parameter transformation group defined by (12) with the infinitesimal generator $\mathbf{v} = \tau(t, u)\partial_t + \xi(t, u)\partial_u$. Assume that $u(a) = 0$, for all admissible functions u. Then G is a variational symmetry group of $\mathcal{L}$ if and only if*

$$\tau \frac{\partial L}{\partial t} + \xi \frac{\partial L}{\partial u} + \left({}_aD_t^\alpha (\xi - \dot{u}\tau) + \left(\frac{d}{dt} {}_aD_t^\alpha u \right) \tau \right) \frac{\partial L}{\partial\, {}_aD_t^\alpha u} + L\dot{\tau} = 0. \tag{18}$$

Proof. See the proof of Theorem 5. $\square$

Example 12. Let $\mathbf{v} = \partial_t + \partial_u$ be the infinitesimal generator of the translation group $(\bar{t}, \bar{u}) = (t + \eta, u + \eta)$. Then $\bar{u}(\bar{t}) = u(\bar{t} - \eta) + \eta$ and

$${}_{\bar{a}}D_{\bar{t}}^\alpha \bar{u} = {}_aD_t^\alpha (u + \eta).$$

3. Nöther's theorem

In the formulation of Nöther's theorem we shall need a generalization of the fractional integration by parts.

Lemma 13. *Let $f, g \in AC([a, b])$. Then, for all $t \in [a, b]$ the following formula holds:*

$$\int_a^t f(s) \cdot {}_sD_b^\alpha g ds = \int_a^t {}_aD_s^\alpha f \cdot g(s) ds + \int_a^t f(s) \cdot \frac{1}{\Gamma(1-\alpha)} \left[\frac{g(b)}{(b-s)^\alpha} - \frac{g(t)}{(t-s)^\alpha} - \int_t^b \frac{\dot{g}(\sigma)}{(\sigma-s)^\alpha} d\sigma \right] ds. \tag{19}$$

Proof. In order to derive (19) we shall use the representation of the right (resp. left) Riemann–Liouville fractional derivative via the right (resp. left) Caputo fractional derivative (4), as well as (3):

$$\begin{aligned}
\int_a^t f(s) \cdot_s D_b^\alpha g ds &= \int_a^t f(s) \cdot \frac{1}{\Gamma(1-\alpha)} \left[\frac{g(b)}{(b-s)^\alpha} - \int_s^b \frac{\dot{g}(\sigma)}{(\sigma-s)^\alpha} d\sigma \right] ds \\
&= \int_a^t f(s) \cdot \frac{1}{\Gamma(1-\alpha)} \left[\frac{g(b)}{(b-s)^\alpha} - \int_s^t \frac{\dot{g}(\sigma)}{(\sigma-s)^\alpha} d\sigma - \int_t^b \frac{\dot{g}(\sigma)}{(\sigma-s)^\alpha} d\sigma \pm \frac{g(t)}{(t-s)^\alpha} \right] ds \\
&= \int_a^t \left[f(s) \cdot_s D_t^\alpha g + f(s) \cdot \frac{1}{\Gamma(1-\alpha)} \left[\frac{g(b)}{(b-s)^\alpha} - \frac{g(t)}{(t-s)^\alpha} - \int_t^b \frac{\dot{g}(\sigma)}{(\sigma-s)^\alpha} d\sigma \right] \right] ds \\
&= \int_a^t \left[{}_aD_s^\alpha f \cdot g(s) + f(s) \cdot \frac{1}{\Gamma(1-\alpha)} \left[\frac{g(b)}{(b-s)^\alpha} - \frac{g(t)}{(t-s)^\alpha} - \int_t^b \frac{\dot{g}(\sigma)}{(\sigma-s)^\alpha} d\sigma \right] \right] ds. \quad \square
\end{aligned}$$

Remark 14. If we put $t = b$ in (19), we obtain (3).

The main goal of symmetry group analysis in the calculus of variations are the first integrals of Euler–Lagrange equations of a variational problem, that is a Nöther-type result.

Analogously to the classical case, an expression $\frac{\mathrm{d}}{\mathrm{d}t}P(t, u(t), {}_a\mathrm{D}_t^\alpha u) = 0$ is called a fractional first integral (or a fractional conservation law) for a fractional differential equation $F(t, u(t), {}_a\mathrm{D}_t^\alpha u) = 0$, if it vanishes along all solutions $u(t)$ of F.

In the statement which is to follow, we prove a version of the fractional Nöther theorem. As we will show, a fractional conserved quantity will also contain integral terms, which is unavoidable, due to the presence of fractional derivatives. If $\alpha = 1$ then (20) reduces to the well-known classical conservation laws for a first-order variational problem.

Theorem 15 (*Nöther's theorem*). *Let G be a local fractional variational symmetry group defined by* (12) *of the fractional variational problem* (5)*, and let* $\mathbf{v} = \tau(t, u(t))\partial_t + \xi(t, u(t))\partial_u$ *be the infinitesimal generator of G. Then*

$$L\tau + \int_a^t \left({}_aD_s^\alpha(\xi - \dot{u}\tau)\frac{\partial L}{\partial_a D_s^\alpha u} - (\xi - \dot{u}\tau)_s D_B^\alpha \left(\frac{\partial L}{\partial_a D_s^\alpha u}\right)\right) \mathrm{d}s = \text{const.} \tag{20}$$

is a fractional first integral (or fractional conservation law) for the Euler–Lagrange equation (9)*.*

The fractional conservation law (20) *can equivalently be written in the form*

$$L\tau - \int_a^t (\xi - \dot{u}\tau)\cdot\frac{1}{\Gamma(1-\alpha)}\left[\frac{\partial_3 L(B, u(B), {}_aD_B^\alpha u)}{(B-s)^\alpha} - \frac{\partial_3 L(t, u(t), {}_a\mathrm{D}_t^\alpha u)}{(t-s)^\alpha} - \int_t^b \frac{\frac{\mathrm{d}}{\mathrm{d}\sigma}\partial_3 L(\sigma, u(\sigma), {}_a D_\sigma^\alpha u)}{(\sigma - s)^\alpha}\mathrm{d}\sigma\right]\mathrm{d}s$$
$$= \text{const.} \tag{21}$$

Remark 16. This theorem is valid in the case when the lower bound a in ${}_a\mathrm{D}_t^\alpha u$ is not transformed, as well as in the case when a is transformed. In the second case we in addition have to suppose that $u(a) = 0$, for all admissible functions u, which provides the same form of the infinitesimal criterion in both cases.

Proof. We want to insert the Euler–Lagrange equation (11) into the infinitesimal criterion (15) or (18). Hence, we will write the latter in a suitable form:

$$\begin{aligned}
0 &= \tau\frac{\partial L}{\partial t} + \xi\frac{\partial L}{\partial u} + \left({}_a\mathrm{D}_t^\alpha(\xi - \dot{u}\tau) + \left(\frac{\mathrm{d}}{\mathrm{d}t}\,{}_a\mathrm{D}_t^\alpha u\right)\tau\right)\frac{\partial L}{\partial\,{}_a\mathrm{D}_t^\alpha u} + L\dot{\tau} \pm \dot{u}\tau\frac{\partial L}{\partial u}\\
&= \tau\left(\frac{\partial L}{\partial t} + \frac{\partial L}{\partial u}\cdot\dot{u} + \frac{\partial L}{\partial\,{}_a\mathrm{D}_t^\alpha u}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\,{}_a\mathrm{D}_t^\alpha u\right) + (\xi - \dot{u}\tau)\frac{\partial L}{\partial u} + {}_a\mathrm{D}_t^\alpha(\xi - \dot{u}\tau)\frac{\partial L}{\partial\,{}_a\mathrm{D}_t^\alpha u} + L\dot{\tau}\\
&= \tau\frac{\mathrm{d}}{\mathrm{d}t}L + L\frac{\mathrm{d}}{\mathrm{d}t}\tau + (\xi - \dot{u}\tau)\frac{\partial L}{\partial u} + {}_a\mathrm{D}_t^\alpha(\xi - \dot{u}\tau)\frac{\partial L}{\partial\,{}_a\mathrm{D}_t^\alpha u} \pm (\xi - \dot{u}\tau)_t D_B^\alpha\left(\frac{\partial L}{\partial\,{}_a\mathrm{D}_t^\alpha u}\right)\\
&= \frac{\mathrm{d}}{\mathrm{d}t}(L\tau) + (\xi - \dot{u}\tau)\left(\frac{\partial L}{\partial u} + {}_tD_B^\alpha\left(\frac{\partial L}{\partial\,{}_a\mathrm{D}_t^\alpha u}\right)\right) + \frac{\mathrm{d}}{\mathrm{d}t}\int_a^t\left({}_aD_s^\alpha(\xi - \dot{u}\tau)\frac{\partial L}{\partial_a D_s^\alpha u} - (\xi - \dot{u}\tau)_s D_B^\alpha\left(\frac{\partial L}{\partial_a D_s^\alpha u}\right)\right)\mathrm{d}s.
\end{aligned}$$

The middle term in the last equation is the Euler–Lagrange equation (11) multiplied by $\xi - \dot{u}\tau$. Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(L\tau + \int_a^t\left({}_aD_s^\alpha(\xi - \dot{u}\tau)\frac{\partial L}{\partial_a D_s^\alpha u} - (\xi - \dot{u}\tau)_s D_B^\alpha\left(\frac{\partial L}{\partial_a D_s^\alpha u}\right)\right)\mathrm{d}s\right) = 0$$

for all solutions of the Euler–Lagrange equation, and (20) holds. If we now apply Lemma 13 we can transform (20) into (21). □

In the next example we consider the case in which the Lagrangian of a fractional variational problem does not depend on t explicitly.

Example 17. Consider a variational problem whose Lagrangian does not depend on t explicitly:

$$\mathcal{L}[u] = \int_A^B L(u(t), {}_a\mathrm{D}_t^\alpha u)\mathrm{d}t. \tag{22}$$

As in the classical case, the one-parameter group of translations of time is a variational symmetry group of (22). Indeed, $\bar{t} = t + \eta$, $\bar{u} = u$ (hence $\bar{u}(\bar{t}) = u(\bar{t} - \eta)$) and ${}_{\bar{a}}D_{\bar{t}}^\alpha\bar{u} = {}_a\mathrm{D}_t^\alpha u$ (see Example 12). Therefore,

$$\int_{\bar{A}}^{\bar{B}} L(\bar{u}(\bar{t}), {}_{\bar{a}}D_{\bar{t}}^\alpha\bar{u})\mathrm{d}\bar{t} = \int_A^B L(u(t), {}_a\mathrm{D}_t^\alpha u)\mathrm{d}t.$$

This fact can be also confirmed by the infinitesimal criterion:

$$\frac{\partial L}{\partial\,{}_a\mathrm{D}_t^\alpha u}\left(-\,{}_a\mathrm{D}_t^\alpha\left(\frac{\mathrm{d}}{\mathrm{d}t}u\right) + \frac{\mathrm{d}}{\mathrm{d}t}\,{}_a\mathrm{D}_t^\alpha u\right) = 0,$$

where the last equality holds since $u(a) = 0$. Thus, $\frac{\mathrm{d}}{\mathrm{d}t}\,{}_a\mathrm{D}_t^\alpha u = {}_a\mathrm{D}_t^\alpha\,\frac{\mathrm{d}}{\mathrm{d}t}u$ (see Introduction).

Using Nöther's theorem we may write a fractional conservation law which comes from this translation group:

$$L + \int_a^t \left(- {}_a\mathrm{D}_t^\alpha \, \dot{u} \cdot \frac{\partial L}{\partial \, {}_a\mathrm{D}_t^\alpha \, u} + \dot{u} \cdot {}_t D_B^\alpha \left(\frac{\partial L}{\partial \, {}_a\mathrm{D}_t^\alpha \, u} \right) \right) \mathrm{d}s = \text{const.}$$

or equivalently

$$L + \int_a^t \dot{u}(s) \cdot \frac{1}{\Gamma(1-\alpha)} \left[\frac{\partial_3 L(u(B), {}_a D_B^\alpha u)}{(B-s)^\alpha} - \frac{\partial_3 L(u(t), {}_a\mathrm{D}_t^\alpha \, u)}{(t-s)^\alpha} - \int_t^b \frac{\frac{\mathrm{d}}{\mathrm{d}\sigma} \partial_3 L(u(\sigma) \, , {}_a D_\sigma^\alpha u)}{(\sigma - s)^\alpha} \mathrm{d}\sigma \right] \mathrm{d}s = \text{const.} \tag{23}$$

Note that (23) tends to

$$L - \dot{u}(t) \cdot \frac{\partial L(u(t), \dot{u}(t))}{\partial \dot{u}} = \text{const.}$$

when $\alpha \to 1^-$, which is the classical energy integral (see [40]).

Example 18. As a concrete example related to Example 17, let us consider a fractional oscillator for which the Lagrangian reads:

$$L = \frac{1}{2} ({}_a\mathrm{D}_t^\alpha \, u)^2 - \omega^2 \frac{u^2}{2}, \tag{24}$$

where ω is a constant (frequency). The fractional variational problem consists of minimizing the functional $\int_0^1 L\mathrm{d}t$, among all smooth functions which satisfy the initial conditions $u(0) = 0$ and $u^{(1)}(0) = 1$. The Euler–Lagrange equation for such an L is:

$$\omega^2 u - {}_t D_1^\alpha \, {}_0 D_t^\alpha u = 0. \tag{25}$$

Since L does not depend on t explicitly and $u(0) = 0$, the translation group $(\bar{t}, \bar{u}) = (t + \eta, u)$ is a fractional variational symmetry group for (24). It generates the following fractional conservation law for (25):

$$\frac{1}{2} ({}_a\mathrm{D}_t^\alpha \, u)^2 - \omega^2 \frac{u^2}{2} + \int_0^t \left(- {}_0 D_s^\alpha u^{(1)} \cdot {}_0 D_s^\alpha u + u^{(1)} \cdot {}_s D_1^\alpha ({}_0 D_s^\alpha u) \right) \mathrm{d}s = \text{const.}$$

4. Approximations

In this section we use approximations of the Riemann–Liouville fractional derivatives by finite sums of derivatives of integer order, which leads to variational problems involving only classical derivatives. We examine the relation between the Euler–Lagrange equations, infinitesimal criterion and Nöther's theorem obtained in the process of approximation and the fractional Euler–Lagrange equations, infinitesimal criterion and Nöther's theorem derived in previous sections.

In the sequel we make the following assumptions:

(i) $L \in \mathcal{C}^N([a, b] \times \mathbb{R} \times \mathbb{R})$, at least, for some $N \in \mathbb{N}$.
(ii) We will simplify the following calculations by considering the case $A = a$ and $B = b$.
(iii) Let (c, d), $-\infty < c < d < +\infty$, be an open interval in $\mathbb{R}$ which contains $[a, b]$, such that for each $t \in [a, b]$ the closed ball $B(t, b - a)$, with center at t and radius $b - a$, lies in (c, d).

In addition we shall separately consider two cases:

(a) Let $u \in \mathcal{C}^\infty([a, b])$ such that $u(a) = a_0$, $u(b) = b_0$, for fixed $a_0, b_0 \in \mathbb{R}$, and L_3 (where L_3 stands for $\partial_3 L$) be a function in $[a, b]$ defined by $t \mapsto L_3(t) = L_3(t, u(t), {}_a\mathrm{D}_t^\alpha \, u)$, $t \in [a, b]$. Let $L_3^{(i)}(b, b_0, p) = 0$, for all $i \in \mathbb{N}$, meaning that for $(t, s, p) \mapsto L_3(t, s, p)$, $t \in [a, b]$, $s, p \in \mathbb{R}$, the following holds:

$$\frac{\partial^i L_3}{\partial t^i}(b, b_0, p) = 0; \qquad \frac{\partial^i L_3}{\partial s^i}(b, b_0, p) = 0; \qquad \frac{\partial^i L_3}{\partial p^i}(b, b_0, p) = 0, \quad \forall p \in \mathbb{R}.$$

(b) Let $u \in \mathcal{C}^\infty([a, b])$ such that $u^{(i)}(b) = 0$, for all $i \in \mathbb{N}_0$, and $u(a) = a_0$, for fixed $a_0 \in \mathbb{R}$. Let $L_3^{(i)}(b) = L_3^{(i)}(b, 0 \, , {}_a D_b^\alpha u) = 0$, for all $i \in \mathbb{N}$ and for every fixed u, meaning that for $(t, s, p) \mapsto L_3(t, s, p)$, $t \in [a, b]$, $s, p \in \mathbb{R}$, the following holds:

$$\frac{\partial^i L_3}{\partial t^i}(b, 0, p) = 0; \qquad \frac{\partial^i L_3}{\partial p^i}(b, 0, p) = 0, \quad \forall p \in \mathbb{R}.$$

Let f be a real analytic function in (c, d). Then according to [41, (15.4) and (1.48)] we have that

$${}_a\mathrm{D}_t^\alpha f = \sum_{i=0}^{\infty} \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)} f^{(i)}(t), \quad t \in B(t, b-a) \subset (c, d), \tag{26}$$

where $\binom{\alpha}{i} = \frac{(-1)^{i-1} \alpha \Gamma(i-\alpha)}{\Gamma(1-\alpha)\Gamma(i+1)}$.

Consider again the fractional variational problem (5). Assume that we are looking for a minimizer $u \in \mathcal{C}^{2N}([a,b])$, for some $N \in \mathbb{N}$. We replace in the Lagrangian the left Riemann–Liouville fractional derivative ${}_a\mathrm{D}_t^\alpha u$ by the finite sum of integer-valued derivatives as in (26):

$$\int_a^b L\left(t, u(t), \sum_{i=0}^{N} \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)} u^{(i)}(t)\right) \mathrm{d}t = \int_a^b \bar{L}(t, u(t), u^{(1)}(t), u^{(2)}(t), \ldots, u^{(N)}(t)) \mathrm{d}t. \tag{27}$$

Now the Lagrangian $\bar{L}$ depends on t, u and all (classical) derivatives of u up to order N. Moreover, $\partial_3\bar{L}, \ldots, \partial_{N+2}\bar{L} \in \mathcal{C}^{N-1}([a,b]\times\mathbb{R}\times\mathbb{R})$, since $\partial_i\bar{L} = \partial_3 L \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}$, $i = 3, \ldots, N+2$. For the problem (27) we can calculate the Euler–Lagrange equations, infinitesimal criterion and conservation laws (via Nöther's theorem).

Now we will consider the fractional variational problem (5) in the case (a) and in the case (b).

The main result of this section is stated in the following theorem.

Theorem 19. *Let $\mathcal{L}[u]$ be a fractional variational problem (5) which is being solved in the case* (a) *or* (b). *Denote by CL the fractional conservation law (20), and by CL_N the fractional conservation law for the Euler–Lagrange equation (28) which corresponds to the variational problem (27). Then*

$CL_N \to CL$ *in the weak sense, as* $N \to \infty$.

Convergence in the weak sense here means that as a test function space we use the space of real analytic functions as follows. (Note that analytic functions are chosen in order to provide convergence of the sum as in (26).)

Let $\mathcal{A}((c,d))$ be the space of real analytic functions in (c,d) with the family of seminorms

$$p_{[m,n]}(\varphi) := \sup_{t\in[m,n]} |\varphi(t)|, \quad \varphi \in \mathcal{A}((c,d)),$$

where $[m,n]$ are subintervals of (c,d). Every function $f \in \mathcal{C}([a,b])$, which we extend to be zero in $(c,d)\setminus[a,b]$, defines an element of the dual $\mathcal{A}'((c,d))$ via

$$\varphi \mapsto \langle f, \varphi\rangle = \int_a^b f(t)\varphi(t)\mathrm{d}t, \quad \varphi \in \mathcal{A}((c,d)).$$

Before we prove the main Theorem 19, we shall prove several auxiliary results.

First we recall a result from [38] which provides an expression for the right Riemann–Liouville fractional derivative in terms of the lower bound a, which figures in the left Riemann–Liouville fractional derivative.

Proposition 20. *Let $F \in \mathcal{C}^\infty([a,b])$, such that $F^{(i)}(b) = 0$, for all $i \in \mathbb{N}_0$, and $F \equiv 0$ in $(c,d)\setminus[a,b]$. Let ${}_t\mathrm{D}_b^\alpha F$ be extended by zero in $(c,d)\setminus[a,b]$. Then:*

(i) *For every $i \in \mathbb{N}$, the $(i-1)$-th derivative of $t \mapsto F(t)(t-a)^{i-\alpha}$ is continuous at $t = a$ and $t = b$ and the i-th derivative of this function, for $i \in \mathbb{N}_0$, is integrable in (c,d) and supported in $[a,b]$.*

(ii) *The partial sums S_N, $N \in \mathbb{N}_0$,*

$$t \mapsto S_N(t) := \begin{cases} \displaystyle\sum_{i=0}^{N} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^i \left(F \cdot \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}\right), & t \in [a,b], \\ 0, & t \in (c,d)\setminus[a,b], \end{cases}$$

are integrable functions in (c,d) supported in $[a,b]$;

(iii)

$${}_t\mathrm{D}_b^\alpha F = \sum_{i=0}^{\infty} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^i \left(F \cdot \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}\right)$$

in the weak sense.

Proof. See [38, Prop. 4.2]. □

The Euler–Lagrange equation for (27) is of the following form:

$$\sum_{i=0}^{N} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^i \frac{\partial\bar{L}}{\partial u^{(i)}} = 0.$$

This is equivalent to

$$\frac{\partial L}{\partial u} + \sum_{i=0}^{N} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^i \left(\partial_3 L \cdot \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}\right) = 0. \tag{28}$$

Remark 21. The Euler–Lagrange equation (28) provides a necessary condition when one solves the variational problem (27) in the class $\mathcal{C}^{2N}([a,b])$, with the prescribed boundary conditions at a and b, i.e., $u(a)=a_0$ and $u(b)=b_0$, a_0, b_0 are fixed real numbers.

The following theorem shows that the Euler–Lagrange equation neqrefeq:EL-N converges to (8), as $N\to+\infty$, in the weak sense. To shorten the notation, we introduce P_N and P for the Euler–Lagrange equations in (8) and (28), respectively.

Theorem 22. *Let $\mathcal{L}[u]$ be the fractional variational problem (5) to be solved in the case (a) or (b). Denote by P the fractional Euler–Lagrange equation (9), and by P_N the Euler–Lagrange equation (28), which correspond to the variational problem (27), in which the left Riemann–Liouville fractional derivative is approximated according to (26) by a finite sum. Then in both cases* (a) *and* (b)

$$P_N \to P \text{ in the weak sense},\quad \text{as } N\to 0.$$

Proof. See [38, Th. 4.3]. □

Let

$$\mathbf{v}_N=\tau_N\frac{\partial}{\partial t}+\xi_N\frac{\partial}{\partial u} \tag{29}$$

be the infinitesimal generator of a local one-parameter variational symmetry group of (27). The N-th prolongation of $\mathbf{v}_N$ is given by

$$\mathrm{pr}^{(N)}\mathbf{v}_N=\mathbf{v}_N+\sum_{i=1}^{N}\xi_N^i\frac{\partial}{\partial u^{(i)}},$$

with

$$\xi_N^i=\frac{\mathrm{d}^i}{\mathrm{d}t^i}\left(\xi_N-\tau_N u^{(1)}\right)+\tau_N u^{(i+1)}.$$

Remark 23. It should be emphasized that the vector field $\mathbf{v}_N$ in (29) differs (in general) from the one introduced at the beginning of Section 2.

Theorem 24. *Let $\mathcal{L}[u]$ and $\bar{\mathcal{L}}[u]$ be fractional variational problems (5) and (27), respectively. Denote by IC and IC_N the corresponding infinitesimal criteria. If $\tau_N\to\tau$ and $\xi_N\to\xi$ as $N\to\infty$, uniformly on compact sets, then*

$$IC_N\to IC \text{ uniformly on compact sets},\quad \text{as } N\to\infty.$$

Proof. The infinitesimal criterion for the variational problem (27) says that a vector field $\mathbf{v}_N$ generates a local one-parameter variational symmetry group of the functional (27) if and only if

$$\mathrm{pr}^{(N)}\mathbf{v}_N(\bar{L})+\bar{L}\dot{\tau}_N=0.$$

This is equivalent to

$$\begin{aligned}
0 &= \tau_N\frac{\partial\bar{L}}{\partial t}+\xi_N\frac{\partial\bar{L}}{\partial u}+\sum_{i=1}^{N}\left(\frac{\mathrm{d}^i}{\mathrm{d}t^i}\left(\xi_N-\tau_N u^{(1)}\right)+\tau_N u^{(i+1)}\right)\frac{\partial\bar{L}}{\partial u^{(i)}}+\bar{L}\dot{\tau}_N\\
&= \tau_N\left(\partial_1 L+\partial_3 L\sum_{i=0}^{N}\binom{\alpha}{i}\frac{(i-\alpha)(t-a)^{i-\alpha-1}}{\Gamma(i+1-\alpha)}u^{(i)}\right)+\xi_N\left(\partial_2 L+\partial_3 L\binom{\alpha}{0}\frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}\right)\\
&\quad+\sum_{i=1}^{N}\left(\frac{\mathrm{d}^i}{\mathrm{d}t^i}\left(\xi_N-\tau_N u^{(1)}\right)+\tau_N u^{(i+1)}\right)\partial_3 L\binom{\alpha}{i}\frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}+L\dot{\tau}_N\\
&= \tau_N\partial_1 L+\xi_N\partial_2 Lv+\left[\sum_{i=1}^{N}\binom{\alpha}{i}\frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}(\xi_N-\tau_N u^{(1)})^{(i)}+\binom{\alpha}{0}\frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}\xi_N\right]\partial_3 L\\
&\quad\pm\binom{\alpha}{0}\frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}\tau_N u^{(1)}+\tau_N\left[\sum_{i=0}^{N}\binom{\alpha}{i}\frac{(i-\alpha)(t-a)^{i-\alpha-1}}{\Gamma(i+1-\alpha)}u^{(i)}+\sum_{i=1}^{N}\binom{\alpha}{i}\frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}u^{(i+1)}\right]\partial_3 L+L\dot{\tau}_N\\
&= \tau_N\partial_1 L+\xi_N\partial_2 L+\left[\sum_{i=0}^{N}\binom{\alpha}{i}\frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}(\xi_N-\tau_N u^{(1)})^{(i)}\right.\\
&\quad\left.+\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=0}^{N}\binom{\alpha}{i}\frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}u^{(i)}\right)\tau_N\right]\partial_3 L+L\dot{\tau}_N.
\end{aligned} \tag{30}$$

So, if $\tau_N \to \tau$ and $\xi_N \to \xi$ as $N \to \infty$, then the last expression on the right hand side tends to

$$\tau\,\partial_1 L + \xi\,\partial_2 L + \left({}_a\mathrm{D}_t^{\alpha}(\xi - \dot{u}\tau) + \left(\frac{\mathrm{d}}{\mathrm{d}t}\, {}_a\mathrm{D}_t^{\alpha}\, u \right) \tau \right) \partial_3 L + L\dot{\tau},$$

and that is exactly the infinitesimal criterion (15) for (5). □

Proof of Theorem 19. First, we will derive the form of a conservation law for the Euler–Lagrange equation (28) of the approximated variational problem (27), which comes from a variational symmetry group with infinitesimal generator (29). For that purpose we need to insert the Euler–Lagrange equation (28) into the infinitesimal criterion (30):

$$\begin{aligned} 0 &= \tau_N \partial_1 L + \xi_N \partial_2 L + \left[\sum_{i=0}^{N} \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)} (\xi_N - \tau_N u^{(1)})^{(i)} + \frac{\mathrm{d}}{\mathrm{d}t} \left(\sum_{i=0}^{N} \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)} u^{(i)} \right) \tau_N \right] \partial_3 L \\ &\quad + L\dot{\tau}_N \pm \tau_N u^{(1)} \partial_2 L = \tau_N \left(\partial_1 L + u^{(1)} \partial_2 L + \partial_3 L \frac{\mathrm{d}}{\mathrm{d}t} \left(\sum_{i=0}^{N} \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)} u^{(i)} \right) \right) + L\dot{\tau}_N + \left(\xi_N - \tau_N u^{(1)} \right) \partial_2 L \\ &\quad + \sum_{i=0}^{N} \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)} (\xi_N - \tau_N u^{(1)})^{(i)} \partial_3 L \pm (\xi_N - \tau_N u^{(1)}) \sum_{i=0}^{N} \left(-\frac{\mathrm{d}}{\mathrm{d}t} \right)^i \left(\binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)} \partial_3 L \right) \\ &= \frac{\mathrm{d}}{\mathrm{d}t} (L\tau_N) + (\xi_N - \tau_N u^{(1)}) \left(\partial_2 L + \sum_{i=0}^{N} \left(-\frac{\mathrm{d}}{\mathrm{d}t} \right)^i \left(\binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)} \partial_3 L \right) \right) \\ &\quad + \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t \left[\sum_{i=0}^{N} \binom{\alpha}{i} \frac{(s-a)^{i-\alpha}}{\Gamma(i+1-\alpha)} (\xi_N - \tau_N u^{(1)})^{(i)} \partial_3 L \right. \\ &\quad \left. - (\xi_N - \tau_N u^{(1)}) \sum_{i=0}^{N} \left(-\frac{\mathrm{d}}{\mathrm{d}s} \right)^i \left(\binom{\alpha}{i} \frac{(s-a)^{i-\alpha}}{\Gamma(i+1-\alpha)} \partial_3 L \right) \right] \mathrm{d}s. \end{aligned}$$

We recognize that the expression in brackets in the middle term in this sum is the Euler–Lagrange equation (28), and hence vanishes. Therefore, we obtain that the following quantity is conserved:

$$\begin{aligned} & L\tau_N + \int_a^t \left[\sum_{i=0}^{N} \binom{\alpha}{i} \frac{(s-a)^{i-\alpha}}{\Gamma(i+1-\alpha)} (\xi_N - \tau_N u^{(1)})^{(i)} \partial_3 L \right. \\ & \left. \quad - (\xi_N - \tau_N u^{(1)}) \sum_{i=0}^{N} \left(-\frac{\mathrm{d}}{ds} \right)^i \left(\binom{\alpha}{i} \frac{(s-a)^{i-\alpha}}{\Gamma(i+1-\alpha)} \partial_3 L \right) \right] \mathrm{d}s = \text{const.} \end{aligned} \tag{31}$$

This of course is one way to write the first integral of the Euler–Lagrange equation (28), which corresponds to the local one-parameter variational symmetry group generated by (29).

Applying Proposition 20(ii) we obtain that (31) converges to (20) in the weak sense, provided that $(\tau_N, \xi_N) \to (\tau, \xi)$, as $N \to \infty$. □

Remark 25. Numerical analysis of fractional differential equations by the use of Theorems 19, 22 and 24 is not established and here we pose an open problem related to the use of our approximation procedure in applications.

5. Concluding remarks

In this paper we studied variational principle containing fractional order derivatives. Especially, we obtained a condition which the symmetry group for this principle has to satisfy. This symmetry group is used for the formulation of a conservation law which generalizes the classical Nöther's theorem. In our work we used left Riemann–Liouville derivatives in Lagrangian density only. An extension including both left and right fractional derivatives could be easily performed. Using a method proposed in our previous work (see [38]) we approximated fractional Euler–Lagrangian equation with finite systems of integer order differential equations. In this way we obtained invariance conditions and corresponding conservation laws for such systems so that these conservation laws converge to the fractional conservation laws when the number of integer order equations tends to infinity.

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