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STABILITY AND BIFURCATION ANALYSIS OF HYPERBOLIC TRAFFIC FLOW MODEL

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Abstract. This paper considers a rather simple hyperbolic model of traffic flow from the point of view of hyperbolic systems of balance laws. The model consists of one conservation law and one balance law with source term, which takes into account time lag in the response of the driver. The following aspects are analyzed, such as (i) genuine coupling of conservative and dissipative part of the system; (ii) shock structure problem and related stability and bifurcation analysis of equilibrium states. The last problem is a matter of ongoing research in the context of hyperbolic systems of balance laws and present study can be regarded as a contribution towards results of more general character.
Keywords: traffic flow, hyperbolic balance laws, stability, bifurcation

1. Introduction

Traffic flow modeling in continuum limit is an old and well established problem [1]. Different kinds of study could be delineated, ranging from mathematical aspects of existence and uniqueness [2] through modeling of traffic flow in kinetic and macroscopic framework to numerical simulation and comparison with experimental data [3].

The aim of this report is to consider a rather simple hyperbolic model of traffic flow from the point of view of hyperbolic systems of balance laws. The model consists of one conservation law and one balance law with source term, which takes into account time lag in the response of the driver. The following aspects will be in focus of this study: (i) genuine coupling of conservative and dissipative part of the system and (ii) shock structure problem and related stability and bifurcation analysis of equilibrium states. Although these issues have the flavor of classic problems, the second one is a matter of ongoing research in the context of hyperbolic systems of balance laws [4]. It will be shown that, in the neighborhood of critical value of shock speed, upstream and downstream equilibria undergo exchange of stability and obey the transcritical bifurcation pattern. These results are in agreement with recent studies of Burgers' equation and isothermal viscoelasticity [5], and pave the way for future results of more general character.

2. Hyperbolic model of traffic flow

In historical perspective, ideas of continuum modeling of traffic flow could be traced back to Lighthill and Whitham [1]. The simplest one-dimensional model is established by a single hyperbolic conservation law

$$\partial_t \rho + \partial_x(\rho V(\rho)) = 0. \tag{1}$$

In (1) ρ denotes density of cars per unit length of the road, $V(\rho)$ is flow velocity, $\rho V(\rho) = q(\rho)$ is density flux and ∂_t and ∂_x denote time and space partial derivatives. This model is built upon single-lane assumption, i.e. the cars are moving in a single lane without taking over. Flow velocity has to be chosen as constitutive function which reasonably fits experimental data. Two approximations which proved to be good are

$$V(\rho) = a(\bar{\rho} - \rho); \quad V(\rho) = a\log(\bar{\rho}/\rho).$$

Possible simple generalizations of this model could be established by taking into account non-local effects like drivers awareness of the conditions ahead of the car and time lag in drivers response. The former assumption leads to the following form of density flux

$$q(\rho) = \rho V(\rho) - \nu \partial_x \rho; \quad \nu > 0, \tag{2}$$

and predicts a diffusion of waves by means of a diffusion-like parabolic model.The latter effect (time lag) can be modeled by another differential equation

$$\partial_t v + v\partial_x v = -\frac{1}{\tau}(v - V(\rho)); \quad \tau > 0, \tag{3}$$

where τ is a measure of driver's response time and has a meaning of relaxation time in the context of hyperbolic systems of balance laws. This equation should be solved together with conservation law

$$\partial_t \rho + \partial_x(\rho v) = 0, \tag{4}$$

so that (3)-(4) form a system of hyperbolic balance laws. Note that for $\tau \to 0$, $v \to V(\rho)$ and the system relaxes to a single conservation law (1).

In this paper, a hyperbolic model of traffic flow which takes into account time lag will be analyzed. It will consist of two equations

$$\begin{aligned} &\partial_t \rho + \partial_x(\rho v) = 0; \\ &\partial_t(\rho v) + \partial_x(\rho v^2 + p(\rho)) = -\frac{1}{\tau}\rho(v - V(\rho)), \end{aligned} \tag{5}$$

and with respect to (3)-(4) contains a generalization in the form of pressure-like term. In such a way it could also be treated as a dissipative generalization of isentropic gas dynamics equations. When $\tau \to 0$ it also relaxes to a conservation law (1).

Since the source term in $(5)_2$ introduces dissipation in the model, it is important to analyze its effects to the whole system. Therefore, two questions will be discussed n this study. The first one will be considered with the structure of hyperbolic model (5) and the way in which dissipation affects its behavior. This will be analyzed using Shizuta-Kawashima condition of genuine coupling. Another one is related to traveling wave solutions which are continuous counterparts of discontinuous shock wave solutions of conservation laws. Their existence will be analyzed through stability and bifurcation analysis of equilibrium states.

3. Genuine coupling

The hyperbolic system (5) could be written in general form

$$\partial_t \mathbf{F}^0(\mathbf{u}) + \partial_x \mathbf{F}(\mathbf{u}) = \mathbf{P}(\mathbf{u}), \tag{6}$$

where $\mathbf{u} = (\rho, v)^T$ is a column-vector of state variables, $\mathbf{F}^0(\mathbf{u})$ and $\mathbf{F}(\mathbf{u})$ are density and flux vectors and $\mathbf{P}(\mathbf{u})$ is the source term. Characteristic speeds Λ of the hyperbolic part of the system are determined as solutions of the eigenvalue problem

$$\det(-\Lambda \mathbf{A}^0 + \mathbf{A}) = 0, \tag{7}$$

where $\mathbf{A}^0 = \partial \mathbf{F}^0 / \partial \mathbf{u}$ and $\mathbf{A} = \partial \mathbf{F} / \partial \mathbf{u}$ are Jacobian matrices. Simple calculation shows that characteristic speeds are

$$\Lambda_1(\rho, v) = v - \sqrt{p'(\rho)}; \quad \Lambda_2(\rho, v) = v + \sqrt{p'(\rho)}. \tag{8}$$

Condition of hyperbolicity, i.e. condition that characteristic speeds are real, imposes the restriction to the structure of pressure-like term

$$p'(\rho) > 0. \tag{9}$$

This condition is well-known from isentropic gas dynamics. For the following analysis it is also needed to calculate the right eigenvectors which correspond to characteristic speeds (8)

$$\mathbf{R}_1 = \begin{pmatrix} -\rho/\sqrt{p'(\rho)} \\ 1 \end{pmatrix}; \quad \mathbf{R}_2 = \begin{pmatrix} \rho/\sqrt{p'(\rho)} \\ 1 \end{pmatrix}. \tag{10}$$

Another restriction to the structure of constitutive functions comes out from the so-called sub-characteristic condition. Namely, for $\mathbf{P}(\mathbf{u}) = \mathbf{0}$ so-called equilibrium manifold could be determined, which relates the non-equilibrium variable v to the equilibrium one ρ. In the model (5) equilibrium manifold is determined by $v_E = V(\rho)$, so that the system is relaxed to a single conservation law (1). This model has its own dynamics which is supposed to be related with (5) in a certain sense. Since the model is hyperbolic, it has its own characteristic speed

$$\lambda(\rho) = V(\rho) + \rho V'(\rho). \tag{11}$$

In order to have dissipative effects genuinely introduced in the system, characteristics of relaxed system must not take over the characteristics of the full hyperbolic system of balance laws. In our problem this is expressed through sub-characteristic condition

$$\Lambda_1(\rho, v_E) < \lambda(\rho) < \Lambda_2(\rho, v_E), \tag{12}$$

which leads to the second structural condition for constitutive functions

$$-\sqrt{p'(\rho)} < \rho V'(\rho) < \sqrt{p'(\rho)}. \tag{13}$$

Genuine coupling condition answers the question about effects of source terms, which are present only in one part of governing equations (not in all of them), to the variables whose behavior is determined by conservation equations. This condition, known in the literature as $K-$condition, was introduced by Shizuta and Kawashima [7]. Its original form can be transformed into condition that eigenvectors of the hyperbolic part of the governing system do not belong to the kernel of Jacobian matrix of source terms $D\mathbf{P} = \partial \mathbf{P} / \partial \mathbf{u}$. In our case it is expressed as $D\mathbf{P} \cdot \mathbf{R}_i \neq \mathbf{0}$, $i = 1, 2$. Since Jacobian matrix of source terms has the form

$$D\mathbf{P} = \begin{pmatrix} 0 & 0 \\ -\frac{1}{\tau}(v - V(\rho) - \rho V'(\rho)) & -\frac{1}{\tau}\rho \end{pmatrix},$$

it can be easily calculated

$$D\mathbf{P}\cdot\mathbf{R}_1 = \begin{pmatrix} 0 \\ -\frac{1}{\tau}\frac{\rho(-v+V(\rho)+\sqrt{p'(\rho)}+\rho V'(\rho))}{\sqrt{p'(\rho)}} \end{pmatrix}; \tag{14}$$

$$D\mathbf{P}\cdot\mathbf{R}_2 = \begin{pmatrix} 0 \\ \frac{1}{\tau}\frac{\rho(-v+V(\rho)-\sqrt{p'(\rho)}+\rho V'(\rho))}{\sqrt{p'(\rho)}} \end{pmatrix}.$$

In view of structural conditions (9) and (13), $K-$condition will be satisfied off the equilibrium manifold for the hyperbolic model of traffic flow (5), and the presence of the source term in balance law $(5)_2$, which governs the behavior of v, will also affect the solutions for the density ρ determined by $(5)_1$.

4. Stability and bifurcation of equilibrium states

This part of the study tends to reveal an interplay between hyperbolic dissipative system (5) and its conservative hyperbolic counterpart (1) through the analysis of shock waves and traveling shock profiles. As it is well-known [8], shock waves are discontinuous weak solutions of conservation laws with jump discontinuities located at the moving surface – the shock front. In one-dimensional case it is reduced to the moving plane front. The simplest possible discontinuous solution is the one which relates two constant states, $\mathbf{u}_+$ in front and $\mathbf{u}_-$ behind the shock wave, while the front moves in the positive $x-$direction with constant speed s. These two states are related to the shock speed by Rankine-Hugoniot condition.

In the problem considered here, conservative hyperbolic system (1) has the following shock wave solution determined by Rankine-Hugoniot equation

$$s(\rho_+ - \rho_-) = \rho_+ V(\rho_+) - \rho_- V(\rho_-). \tag{15}$$

This shock wave solution is admissible if the shock speed s satisfies Lax admissibility condition $\lambda(\rho_+) < s < \lambda(\rho_-)$, which for (11) reads

$$V(\rho_+) + \rho_+ V'(\rho_+) < s < V(\rho_-) + \rho_- V'(\rho_-). \tag{16}$$

This condition introduces irreversibility in weak solutions of conservation laws since the states in front and behind the shock cannot be interchanged.

The main question of this analysis is how does the dissipative source term in $(5)_2$ affect the shock wave solution of the conservation law (1)? The same question was posed in [4, 5] and this paper will follow the same steps of analysis performed there. The first one if transformation of the model (5) into a system of ordinary differential equations using traveling wave ansatz $\mathbf{u} = \hat{\mathbf{u}}(\xi)$, $\xi = x - st$. This assumption leads to the following system

$$-s\frac{d\rho}{d\xi} + \frac{d}{d\xi}(\rho v) = 0; \tag{17}$$

$$-s\frac{d}{d\xi}(\rho v) + \frac{d}{d\xi}(\rho v^2 + p(\rho)) = -\frac{1}{\tau}\rho(v - V(\rho)).$$

It is also assumed that traveling wave solution asymptotically connects upstream equilibrium state $(\rho_+, V(\rho_+))$ and downstream equilibrium state $(\rho_-, V(\rho_-))$. Equation $(17)_1$ can be integrated to obtain

$$-s(\rho - \rho_-) + (\rho v - \rho_- V(\rho_-)) = 0, \tag{18}$$

where downstream equilibrium state $\rho(-\infty) = \rho_-$, $v(-\infty) = v_E(-\infty) = V(\rho_-)$ is used in the course of integration. By introducing (18) into $(17)_2$ a single equation for the shock profile is obtained

$$\dot{\rho} = \phi(\rho, s) = -\frac{1}{\tau} \frac{\rho_- V(\rho_-) - \rho V(\rho) - s(\rho - \rho_-)}{p'(\rho) - (s - (1/\rho)(\rho_- V(\rho_-) - s(\rho - \rho_-)))^2}, \tag{19}$$

where and overdot stands for derivative with respect to ξ. It is important to note that equilibrium states ρ_- and ρ_+ are stationary points of (19), i.e. $\phi(\rho_-, s) = \phi(\rho_+, s) = 0$ due to (15).

The second step is stability analysis of equilibrium state ρ_-. It is based upon linear variational equation which reads

$$\dot{y} = \Phi(\rho_-, s)y; \tag{20}$$
$$\Phi(\rho_-, s) = \phi_\rho(\rho_-, s) = -\tau^{-1} \frac{s - V(\rho_-) - \rho_- V'(\rho_-)}{p'(\rho_-) - (s - V(\rho_-))^2},$$

for perturbation $y = \rho - \rho_-$. Stability of equilibrium state is determined by the sign of $\Phi(\rho_-, s)$. Simple observation shows that there exists a critical value of the shock speed s^* for which the equilibrium state could change its stability:

$$s^* = V(\rho_-) + \rho_- V'(\rho_-) \tag{21}$$

The sign of $\Phi(\rho_-, s)$ depends also on the sign of denominator. By combining (13) and (16) it can be found out that the following inequality holds for the admissible shocks

$$s - V(\rho_-) < \rho_- V'(\rho_-) < \sqrt{p'(\rho_-)},$$

and the denominator is positive. If the shock is not admissible, i.e. $s - V(\rho_-) > \rho_- V'(\rho_-)$, the sign of denominator, although not so obvious, can be determined by continuity argument. Namely, due to strict inequality $\rho_- V'(\rho_-) < \sqrt{p'(\rho_-)}$ there exists a neighborhood of the critical value s^*, $s \in (s^*, \bar{s})$ such that

$$s - V(\rho_-) < \sqrt{p'(\rho_-)}$$

holds and the denominator is still positive. Therefore, the following conclusion about stability of downstream equilibrium can be derived:

$$\begin{array}{lll} s < s^* & \Rightarrow & (\rho_-, V(\rho_-)) \text{ – unstable;} \\ s > s^* & \Rightarrow & (\rho_-, V(\rho_-)) \text{ – stable.} \end{array} \tag{22}$$

The most important feature of this result is that exchange of stability of equilibrium state occurs in the neighborhood of the critical value $s^* = \lambda(\rho_-)$ which coincides with the characteristic speed of relaxed conservation law (1), rather than any characteristic speed of the full hyperbolic model (5). In such a way stability condition (22) coincides with Lax criterion and can be regarded as inherent selection rule for admissible shock profiles just as Lax condition for shock waves.

In order to support these assertions it will be shown that (ρ_-, s^*) represents a bifurcation point of downstream equilibrium state as a stationary point of (19). Expansion of $\phi(\rho, s)$ in Taylor series in the neighborhood of (ρ_-, s^*) up to second order shows that

$$\dot{y} \approx \frac{1}{\tau} \frac{-\mu y + (2V'(\rho_-) + \rho_- V''(\rho_-))y^2}{p'(\rho_-) - (\rho_- V'(\rho_-))^2} \tag{23}$$

for $\mu = s - s^*$. It is inevitable that this equation describes transcritical bifurcation pattern which is typical for shock profiles which smooth out shock waves corresponding to genuinely nonlinear characteristic speed. Similar results have been observed for Burgers' equation and isothermal viscoelasticity [5], while complete gas dynamics equations with viscosity and heat conduction called for reduction procedure [4].

5. Conclusions

This study provided a contribution related to shock profile analysis of hyperbolic systems through stability an bifurcation analysis of equilibrium state. It was shown that hyperbolic model for traffic flow (5) obeys the so-called $K-$condition of genuine coupling. Consequently, dissipative terms which are present only in one part of governing equations affect the whole system. Furthermore, it was shown that downstream equilibrium changes its stability when shock speed crosses the critical vale which coincides with the characteristic speed of relaxed conservative system. It also obeys transcritical bifurcation pattern leading to the admissible shock profile equation.

It is a matter of ongoing research to establish selection rule, in the spirit of stability and bifurcation analysis, for admissible shock profiles of hyperbolic systems of balance laws.

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References

[1] Lighthill M J, Whitham G B (1955) On Kinematic Waves. II. A Theory of Traśc Flow on Long Crowded Roads, *Proceedings of the Royal Society of London, Series A*, **229**, pp. 317–345.

[2] Illner R et al. (1999), A Kinetic Model for Vehicular Traffic: Existence of Stationary Solutions, *Journal of Mathematical Analysis and Applications*, **237**, pp. 622–643.

[3] Kerner B S (2000) Experimental features of the emergence of moving jams in free traffic flow, *Journal of Physics A: Mathematical and General*, **33**, pp. L221–L228.

[4] Simić S S (2009) Shock structure in continuum models of gas dynamics: stability and bifurcation analysis, *Nonlinearity*, **22**(6), pp 1337–1366.

[5] Simić S S (2008) A Note on Shock Profles in Dissipative Hyperbolic and Parabolic Models, *Publications de l'Institut Mathématique*, **84**(98), pp. 97–107.

[6] Whitham G B (1974) *Linear and Nonlinear Waves*, John Wiley and Sons, New York.

[7] Shizuta Y, Kawashima S (1985) Systems of equations of hyperbolic-parabolic type with applications to the discrete Boltzmann equation, textitHokkaido Math. J., **14**, pp. 249–275.

[8] Dafermos C M (2000) *Hyperbolic Conservation Laws in Continuum Physics*, Springer-Verlag, Berlin.