

# On the hyperbolic system of a mixture of Eulerian fluids: A comparison between single- and multi-temperature models

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## SUMMARY

The first rational model of homogeneous mixtures of fluids was proposed by Truesdell in the context of rational thermodynamics. Afterwards, two different theories were developed: one with a single-temperature (ST) field of the mixture and the other one with several temperatures. The two systems are from the mathematical point of view completely different and the relationship between their solutions was not clarified.

In this paper, the hyperbolic multi-temperature (MT) system of a mixture of Eulerian fluids will be explained and it will be shown that the corresponding single-temperature differential system is a *principal subsystem* of the MT one. As a consequence, the subcharacteristic conditions for characteristic speeds hold and this gives an upper-bound esteem for pulse speeds in an ST model. Global behaviour of smooth solutions for large time for both systems will also be discussed through the application of the Shizuta–Kawashima condition. Finally, as an application, the particular case of a binary mixture is considered. Copyright © 2006 John Wiley & Sons, Ltd.

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## 1. INTRODUCTION

In the mixture theory of fluids, different models could be proposed based either on continuum theory or on kinetic theory of gases. Both approaches suggest different ideas with the aim to give

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an appropriate explanation to irreversible phenomena like diffusion, heat transfer and chemical reactions. Among these unifying attempts appeared two constitutive theories with almost independent paths. They branched out from the question whether the constituents of the mixture should have a common temperature or not. Both of these approaches, i.e. single-temperature (ST) and multi-temperature (MT) ones, gained considerable successes in modelling the behaviour of the mixture.

The first rational model of homogeneous mixtures was proposed by Truesdell in the context of rational thermodynamics [1]. A mathematical model of a mixture with multiple temperatures was carefully developed within the framework of continuum thermomechanics. In the same context, a single-temperature model was derived, albeit independently of an MT one and with strikingly different consequences, in particular, concerning the entropy principle and wave propagation (see e.g. the monographs [2, 3] and the references therein).

The MT approach is naturally embedded into Maxwell's kinetic theory of mixtures as can be seen in [4, 5]. This theory comes on its own especially when atomic masses of the constituents are different or in plasmas where the constituents are electrons, ions and neutral atoms. The relevance of the MT model is thus put in evidence and can be further supported by the analysis of plasma behaviour at high temperatures, thoroughly performed by Kannappan and Bose [6, 7] and Bose and Seeniraj [8].

Apart from physical reasons, these two theories are completely different also in mathematical sense: in general, the MT system does not admit the solution with equal temperature  $T_1 = T_2 = \dots = T$  also if we require this condition initially.

The aim of this study is to describe the relationships between these two models—in the case of Euler's fluids—and to show that ST model, within the framework of hyperbolic systems, represents a *principal subsystem* of the MT model in the sense of a definition due to Boillat and Ruggeri [9]. This implies a particular link between the two theories and we are able to discuss the subcharacteristic conditions and the asymptotic behaviour of the smooth solutions.

To achieve this goal, several steps have to be made. Firstly, in Section 2, the general structure of MT model will be given, including the study of restrictions imposed on the constitutive functions by the objectivity principle (Galilean invariance), the entropy principle and the entropy convexity. This will be needed for the proof, given in Section 3, that ST model represents a principal subsystem of an MT one. As a consequence, the characteristic speeds of the ST model satisfy subcharacteristic conditions, i.e. they are bounded by the greatest and the smallest characteristic speeds of the MT model.

In Section 4, we present a qualitative analysis with particular emphasis on the existence of global smooth solutions for initial data that are perturbations of equilibrium states. It will be shown that MT model satisfies the Shizuta–Kawashima condition [10]. As a consequence, when initial data are smooth and sufficiently small, a global smooth solution does exist for every  $t$  and converges to a constant equilibrium state. Instead, the ST model without chemical reactions obeys neither this condition nor its weak form [11] and we can expect a blow-up of the solutions even for small initial data. Thanks to these results, we can establish a hierarchy in which ST model appears as an approximation of MT approach and it can be expected that the solution of MT model will converge to the solution of ST model before it reaches an equilibrium state.

As an application, a particular case of a binary mixture of Euler fluids will be discussed in Section 5 in which governing equations will be rewritten in the form of a single fluid with structure, following the idea of Ruggeri in the ST case [12].

2. THE MIXTURE THEORY

2.1. The field equations

The description of simple homogeneous mixtures in the context of rational thermodynamics [1] relies on the postulate that each constituent obeys the same balance laws as a single fluid. They express rates of change of mass, momentum and energy with appropriate production terms due to mutual interaction of the constituents:

$$\begin{aligned} \frac{\partial \rho_\alpha}{\partial t} + \operatorname{div}(\rho_\alpha \mathbf{v}_\alpha) &= \tau_\alpha \\ \frac{\partial(\rho_\alpha \mathbf{v}_\alpha)}{\partial t} + \operatorname{div}(\rho_\alpha \mathbf{v}_\alpha \otimes \mathbf{v}_\alpha - \mathbf{t}_\alpha) &= \mathbf{m}_\alpha \quad (\alpha = 1, 2, \dots, n) \quad (1) \\ \frac{\partial(\frac{1}{2}\rho_\alpha v_\alpha^2 + \rho_\alpha \varepsilon_\alpha)}{\partial t} + \operatorname{div} \left\{ \left( \frac{1}{2}\rho_\alpha v_\alpha^2 + \rho_\alpha \varepsilon_\alpha \right) \mathbf{v}_\alpha - \mathbf{t}_\alpha \mathbf{v}_\alpha + \mathbf{q}_\alpha \right\} &= e_\alpha \end{aligned}$$

$\rho_\alpha$  being the density,  $\mathbf{v}_\alpha$  the velocity,  $\varepsilon_\alpha$  the internal energy,  $\mathbf{t}_\alpha$  the stress tensor and  $\mathbf{q}_\alpha$  the heat flux of the  $\alpha$  constituent of the mixture. The production terms must satisfy the following relations:

$$\sum_{\alpha=1}^n \tau_\alpha = 0, \quad \sum_{\alpha=1}^n \mathbf{m}_\alpha = \mathbf{0}, \quad \sum_{\alpha=1}^n e_\alpha = 0 \quad (2)$$

due to conservation of mass, momentum and energy of the mixture.

In fact, one may obtain from (1) the conservation laws of total mass, total momentum and total energy of the mixture through summation of the corresponding balance laws (1), after the introduction of following quantities:

$$\begin{aligned} \rho &= \sum_{\alpha=1}^n \rho_\alpha \quad \text{total mass density} \\ \mathbf{v} &= \frac{1}{\rho} \sum_{\alpha=1}^n \rho_\alpha \mathbf{v}_\alpha \quad \text{mixture velocity} \\ \mathbf{u}_\alpha &= \mathbf{v}_\alpha - \mathbf{v} \quad \left( \sum_{\alpha=1}^n \rho_\alpha \mathbf{u}_\alpha = \mathbf{0} \right) \quad \text{diffusion velocity} \\ \mathbf{t} &= \sum_{\alpha=1}^n (\mathbf{t}_\alpha - \rho_\alpha \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha) \quad \text{stress tensor} \\ \varepsilon &= \frac{1}{\rho} \sum_{\alpha=1}^n \rho_\alpha \left( \varepsilon_\alpha + \frac{1}{2} u_\alpha^2 \right) \quad \text{internal energy} \\ \mathbf{q} &= \sum_{\alpha=1}^n \left\{ \mathbf{q}_\alpha + \rho_\alpha \left( \varepsilon_\alpha + \frac{1}{2} u_\alpha^2 \right) \mathbf{u}_\alpha - \mathbf{t}_\alpha \mathbf{u}_\alpha \right\} \quad \text{flux of internal energy} \end{aligned} \quad (3)$$

The conservation laws read as the one of a single fluid:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) &= 0 \\ \frac{\partial(\rho \mathbf{v})}{\partial t} + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v} - \mathbf{t}) &= \mathbf{0} \\ \frac{\partial(\frac{1}{2}\rho v^2 + \rho \varepsilon)}{\partial t} + \operatorname{div} \left\{ \left( \frac{1}{2}\rho v^2 + \rho \varepsilon \right) \mathbf{v} - \mathbf{t} \mathbf{v} + \mathbf{q} \right\} &= 0 \end{aligned} \quad (4)$$

In ordinary thermodynamics of mixtures, the state of the mixture is determined by  $n + 4$  fields, the mass densities  $\rho_\alpha$ , the mixture velocity  $\mathbf{v}$  and the common temperature  $T$ . Therefore,  $n + 4$  field equations have to be chosen to describe thermodynamic processes. These are balance laws of mass (1)<sub>1</sub> and conservation laws of momentum and energy (4)<sub>2,3</sub>. Usually, one of the balance laws (1)<sub>1</sub> is replaced by the conservation law (4)<sub>1</sub>. Consequently, one may choose  $\rho$ ,  $\mathbf{v}$ ,  $T$  and  $n - 1$  independent concentration variables  $c_\alpha$  as field variables, where

$$c_\alpha = \frac{\rho_\alpha}{\rho}, \quad \sum_{\alpha=1}^n c_\alpha = 1 \quad (5)$$

This model is characterized by an important shortcoming due to its constitutive assumptions (see [3, Chapter 5]): it predicts infinite speed of propagation of disturbances in concentration also when viscosity and heat conductivity are neglected. As it will be seen, this problem has been successfully solved.

Rational extended thermodynamics [3] describes thermodynamic processes by the  $4n + 1$  fields of mass densities  $\rho_\alpha$ , constituent velocities  $\mathbf{v}_\alpha$  and common temperature  $T$ . The field equations are thus the balance laws of mass and momentum (1)<sub>1,2</sub> and the conservation law of energy (4)<sub>3</sub>. Like in the previous case, conservation laws of mass and momentum (4)<sub>1,2</sub> may replace the corresponding balance laws for one constituent. Thus, one may choose  $\rho$ ,  $\mathbf{v}$ ,  $T$ , and  $n - 1$  independent concentrations  $c_\alpha$  and diffusion velocities  $\mathbf{u}_\alpha$  as field variables. Finally, one may consider a model of a mixture in which each constituent has its own temperature. This assumption leads to an MT model of the mixture. In such a situation, we have  $5n$  variables  $\rho_\alpha$ ,  $\mathbf{v}_\alpha$  and  $T_\alpha$ , and all the  $5n$  balance laws (1) are needed for the description of the thermodynamic processes. Using the same idea as in previous cases, balance laws for one of the constituents, say  $n$ , could be replaced by conservation laws (4). Adequate field variables are the density, velocity and internal energy of the mixture ( $\rho$ ,  $\mathbf{v}$ ,  $\varepsilon$ ) and  $n - 1$  independent concentrations, diffusion velocities and internal energies for the constituents ( $c_\alpha$ ,  $\mathbf{u}_\alpha$ ,  $\varepsilon_\alpha$ ).

In this study an MT approach will be adopted and the following vector of field variables will be used:<sup>§</sup>

$$\mathbf{u} = (\rho, \mathbf{v}, \varepsilon, \rho_b, \mathbf{v}_b, \varepsilon_b)^T, \quad b = 1, \dots, n - 1 \quad (6)$$

<sup>§</sup>Also in the following, the index  $\alpha$  runs from 1 to  $n$ , while italic  $a$  or  $b$  runs from 1 to  $n - 1$ .

With this choice, the governing equations for the  $n$ -component mixture read

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) &= 0 \\ \frac{\partial(\rho \mathbf{v})}{\partial t} + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v} - \mathbf{t}) &= \mathbf{0} \\ \frac{\partial(\frac{1}{2}\rho v^2 + \rho \varepsilon)}{\partial t} + \operatorname{div} \left\{ \left( \frac{1}{2}\rho v^2 + \rho \varepsilon \right) \mathbf{v} - \mathbf{t} \mathbf{v} + \mathbf{q} \right\} &= 0 \\ \frac{\partial \rho_b}{\partial t} + \operatorname{div}(\rho_b \mathbf{v}_b) &= \tau_b \\ \frac{\partial(\rho_b \mathbf{v}_b)}{\partial t} + \operatorname{div}(\rho_b \mathbf{v}_b \otimes \mathbf{v}_b - \mathbf{t}_b) &= \mathbf{m}_b \\ \frac{\partial(\frac{1}{2}\rho_b v_b^2 + \rho_b \varepsilon_b)}{\partial t} + \operatorname{div} \left\{ \left( \frac{1}{2}\rho_b v_b^2 + \rho_b \varepsilon_b \right) \mathbf{v}_b - \mathbf{t}_b \mathbf{v}_b + \mathbf{q}_b \right\} &= e_b, \quad b = 1, \dots, n-1 \end{aligned} \tag{7}$$

2.2. Galilean invariance of field equations

System (7) is a particular case of the balance law one:

$$\partial_t \mathbf{F}^0 + \partial_i \mathbf{F}^i = \mathbf{F} \tag{8}$$

where  $\mathbf{F}^0$  (densities),  $\mathbf{F}^i$  (fluxes) and  $\mathbf{F}$  (productions) are  $N$ -vectors functions of the field variables. It is convenient to divide field variables into a pair  $\mathbf{u} = (\mathbf{w}, \mathbf{v})$ ,  $\mathbf{v}$  being the velocity field and  $\mathbf{w}$  objective quantities. Moreover, the non-convective fluxes can be defined as  $\mathbf{G}^i = \mathbf{F}^i - \mathbf{F}^0 v^i$ , thus putting Equation (8) in the form

$$\partial_t \mathbf{F}^0 + \partial_i (\mathbf{F}^0 v^i + \mathbf{G}^i) = \mathbf{F} \tag{9}$$

Principle of relativity requires that field equations are invariant with respect to Galilean transformations:

$$x^i \mapsto x^{i*} - c^i t^*, \quad t \mapsto t^*, \quad v^i \mapsto v^{i*} - c^i \tag{10}$$

i.e. system (9) has the same form in the new frame:

$$\partial_{t^*} \mathbf{F}^0(\mathbf{w}, \mathbf{v}^*) + \partial_{i^*} (\mathbf{F}^0(\mathbf{w}, \mathbf{v}^*) v^{i^*} + \mathbf{G}^i(\mathbf{w}, \mathbf{v}^*)) = \mathbf{F}(\mathbf{w}, \mathbf{v}^*) \tag{11}$$

As a consequence, (see Ruggeri [13]), there exists a linear operator  $\mathbf{X}(\mathbf{v})$  such that

$$\begin{aligned} \mathbf{F}^0(\mathbf{w}, \mathbf{v}) &= \mathbf{X}(\mathbf{v}) \hat{\mathbf{F}}^0(\mathbf{w}) \\ \mathbf{G}^i(\mathbf{w}, \mathbf{v}) &= \mathbf{X}(\mathbf{v}) \hat{\mathbf{G}}^i(\mathbf{w}) \\ \mathbf{F}(\mathbf{w}, \mathbf{v}) &= \mathbf{X}(\mathbf{v}) \hat{\mathbf{F}}(\mathbf{w}) \end{aligned} \tag{12}$$

where

$$\hat{\mathbf{F}}^0(\mathbf{w}) = \mathbf{F}^0(\mathbf{w}, \mathbf{0}), \quad \hat{\mathbf{G}}^i(\mathbf{w}) = \mathbf{G}^i(\mathbf{w}, \mathbf{0}), \quad \hat{\mathbf{F}}(\mathbf{w}) = \mathbf{F}(\mathbf{w}, \mathbf{0})$$

are objective ‘internal’ densities, fluxes and productions, while  $\mathbf{X}(\mathbf{v})$  has the following properties:

$$\mathbf{X}(\mathbf{a} + \mathbf{b}) = \mathbf{X}(\mathbf{a})\mathbf{X}(\mathbf{b}) = \mathbf{X}(\mathbf{b})\mathbf{X}(\mathbf{a}), \quad \mathbf{X}(\mathbf{0}) = \mathbf{I}$$

i.e.

$$\mathbf{X}(\mathbf{v}) = e^{\mathbf{A}^r v_r}$$

with  $\mathbf{A}^r$  constant  $(3 \times 3)$  matrix such that  $\mathbf{A}^r \mathbf{A}^s = \mathbf{A}^s \mathbf{A}^r$ ,  $(r, s = 1, 2, 3)$ .

From (7), using notation (9) one can recognize densities, fluxes and productions in component-wise form, along with their internal counterparts:

$$\mathbf{F}^0 = \begin{pmatrix} \rho \\ \rho v^j \\ \frac{1}{2}\rho v^2 + \rho \varepsilon \\ \rho c_b \\ \rho c_b (u_b^j + v^j) \\ \frac{1}{2}\rho c_b (\mathbf{u}_b + \mathbf{v})^2 + \rho c_b \varepsilon_b \end{pmatrix}, \quad \hat{\mathbf{F}}^0 = \begin{pmatrix} \rho \\ 0^k \\ \rho \varepsilon \\ \rho c_b \\ \rho c_b u_b^k \\ \frac{1}{2}\rho c_b u_b^2 + \rho c_b \varepsilon_b \end{pmatrix} \quad (13)$$

$$\mathbf{G}^i = \begin{pmatrix} 0^i \\ -t^{ij} \\ -t^{ik} v_k + q^i \\ \rho c_b u_b^i \\ \rho c_b u_b^i (u_b^j + v^j) - t_b^{ij} \\ \left\{ \begin{array}{l} (\frac{1}{2}\rho c_b (\mathbf{u}_b + \mathbf{v})^2 + \rho c_b \varepsilon_b) u_b^i \\ -t_b^{ik} (u_b^k + v_k) + q_b^i \end{array} \right\} \end{pmatrix}, \quad \hat{\mathbf{G}}^i = \begin{pmatrix} 0^i \\ -t^{ik} \\ q^i \\ \rho c_b u_b^i \\ \rho c_b u_b^i u_b^k - t_b^{ik} \\ \left\{ \begin{array}{l} (\frac{1}{2}\rho c_b u_b^2 + \rho c_b \varepsilon_b) u_b^i \\ -t_b^{ik} u_b^k + q_b^i \end{array} \right\} \end{pmatrix} \quad (14)$$

$$\mathbf{F} = \begin{pmatrix} 0 \\ 0^j \\ 0 \\ \tau_b \\ m_b^j \\ e_b \end{pmatrix}, \quad \hat{\mathbf{F}} = \begin{pmatrix} 0 \\ 0^k \\ 0 \\ \hat{\tau}_b \\ \hat{m}_b^k \\ \hat{e}_b \end{pmatrix} \quad (15)$$

*Statement 1*

The linear operator  $\mathbf{X}(\mathbf{v})$  which assures Galilean invariance of field equations (7) has the form

$$\mathbf{X}(\mathbf{v}) = \begin{pmatrix} 1 & 0_k & 0 & 0 & 0_k & 0 \\ v^j & \delta^{jk} & 0^j & 0^j & 0^{jk} & 0^j \\ \frac{v^2}{2} & v_k & 1 & 0 & 0_k & 0 \\ 0 & 0_k & 0 & 1 & 0_k & 0 \\ 0^j & 0^{jk} & 0^j & v^j & \delta^{jk} & 0^j \\ 0 & 0_k & 0 & \frac{v^2}{2} & v_k & 1 \end{pmatrix} \tag{16}$$

As a consequence, the following relations between the production terms and their internal counterparts are obtained:

$$\begin{aligned} \tau_b &= \hat{\tau}_b \\ m_b^j &= \hat{\tau}_b v^j + \hat{m}_b^j \\ e_b &= \hat{\tau}_b \frac{v^2}{2} + \hat{m}_b^k v_k + \hat{e}_b \end{aligned} \tag{17}$$

In (16)  $0^i$ ,  $0_i$  and  $0^{ik}$  indicate, respectively, the zero column vector ( $3 \times 1$ ), the zero row vector ( $1 \times 3$ ), and the ( $3 \times 3$ ) null matrix,  $\delta^{jk}$  is the Kronecker delta and only in this case we have difference between  $v_k$  and  $v^j$  that indicates the velocity components in row or in column, respectively.

The first part of the statement can be proved by direct application of relations (12)<sub>1,2</sub> to densities (13) and fluxes (14). Once the operator  $\mathbf{X}(\mathbf{v})$  is determined, the second part of the statement can be derived from (12)<sub>3</sub> and (15).

2.3. Euler fluids

Up to now, it was not necessary to introduce the constitutive equations that are needed for system (7) in order to close the system of PDE. Now, let us introduce the assumption that all the constituents of the mixture are Eulerian fluids, i.e. neither viscous nor heat conducting, i.e.

$$\mathbf{t}_\alpha = -p_\alpha \mathbf{I}, \quad \mathbf{q}_\alpha = \mathbf{0} \quad (\alpha = 1, \dots, n) \tag{18}$$

As a consequence of this assumption, the stress tensor and the heat flux (3) are reduced to

$$\begin{aligned} \mathbf{t} &= -p \mathbf{I} - \sum_{\alpha=1}^n (\rho_\alpha \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha) \\ \mathbf{q} &= \sum_{\alpha=1}^n \left\{ \rho_\alpha \left( \varepsilon_\alpha + \frac{1}{2} u_\alpha^2 \right) + p_\alpha \right\} \mathbf{u}_\alpha \end{aligned} \tag{19}$$

where  $p_\alpha$  is the pressure of the  $\alpha$ -constituent and

$$p = \sum_{\alpha=1}^n p_\alpha \tag{20}$$

is the total pressure.

#### 2.4. The entropy principle and its restrictions

The existence of the linear operator (16) confirms Galilean invariance of field equations (7) and determines the velocity dependence of the production terms (17). Another important restriction comes from the entropy inequality, i.e. the supplementary balance law

$$\partial_t h^0 + \partial_i (h^0 v^i + \varphi^i) = \Sigma \leq 0 \quad (21)$$

where  $-h^0$ ,  $-\varphi^i$  and  $-\Sigma$  are the entropy density, the non-convective entropy flux and the entropy production, respectively. Like field equations, the supplementary balance law (21) must also be invariant with respect to Galilean transformations. Therefore, density, flux and production of entropy have the following form [13]:<sup>‡</sup>

$$h^0 = h^0(\mathbf{w}), \quad \varphi^i = \varphi^i(\mathbf{w}), \quad \Sigma = \Sigma(\mathbf{w})$$

Since both balance laws (8) and entropy inequality (21) are quasi-linear first-order partial differential equations (PDEs), their compatibility can be achieved through the use of the *main field*  $\mathbf{u}'(\mathbf{u})$  (see Boillat [14, 15] and Ruggeri and Strumia [16]):

$$\partial_t h^0 + \partial_i h^i - \Sigma \equiv \mathbf{u}' \cdot \{ \partial_t \mathbf{F}^0 + \partial_i \mathbf{F}^i - \mathbf{F} \} \quad (22)$$

where  $h^i = h^0 v^i + \varphi^i$ . As a consequence, the following relations hold:

$$dh^0 = \mathbf{u}' \cdot d\mathbf{F}^0, \quad dh^i = \mathbf{u}' \cdot d\mathbf{F}^i, \quad \Sigma = \mathbf{u}' \cdot \mathbf{F} \leq 0, \quad (i = 1, 2, 3) \quad (23)$$

When the entropy density  $h^0$  is a convex function of  $\mathbf{u} \equiv \mathbf{F}^0$ , one may use from (23)<sub>1</sub> its gradients as field variables

$$\mathbf{u}' = \frac{\partial h^0}{\partial \mathbf{u}}$$

so that original balance laws could be transformed in a symmetric form. In fact, introducing the four potentials [14]

$$h'^0 = \mathbf{u}' \cdot \mathbf{F}^0 - h^0, \quad h'^i = \mathbf{u}' \cdot \mathbf{F}^i - h^i \quad (i = 1, 2, 3) \quad (24)$$

the original system can be put in the special symmetric form

$$\partial_t \left( \frac{\partial h'^0}{\partial \mathbf{u}'} \right) + \partial_i \left( \frac{\partial h'^i}{\partial \mathbf{u}'} \right) = \mathbf{F} \quad (25)$$

$$\iff$$

$$\frac{\partial^2 h'^0}{\partial \mathbf{u}' \partial \mathbf{u}'} \partial_t \mathbf{u}' + \frac{\partial^2 h'^i}{\partial \mathbf{u}' \partial \mathbf{u}'} \partial_i \mathbf{u}' = \mathbf{F} \quad (i = 1, 2, 3)$$

The Boillat proof [14, 15] has the advantage with respect to the symmetrization of Friedrichs and Lax [17] that the symmetric system is the original one. Moreover this includes, as a particular

<sup>‡</sup>Also in the following, we adopt the convention typical in the mathematical community of hyperbolic PDEs to call  $h^0$ ,  $h^i = h^0 v^i + \varphi^i$  and  $\Sigma$ , entropy density, entropy flux and entropy production. Of course, the physical ones are the same except for the sign.



case, the example discovered first by Godunov for the fluid dynamics case and the Euler–Lagrange systems [18]. Symmetric structure of the system of balance laws is highly desirable due to hyperbolicity and local well posedness of initial-value problems (see e.g. [19]). Moreover, convexity of  $h^0$  guarantees also the thermodynamic stability. In the present case,

$$h^0 = -\rho S = -\sum_{\alpha=1}^n \rho_{\alpha} S_{\alpha} \tag{26}$$

where  $S$  is the total entropy density and  $S_{\alpha}$  are the entropy densities of each constituent.

*Statement 2*

The entropy density  $h^0(26)$  of the mixture is a convex function with respect to the densities  $\mathbf{u} \equiv (\rho_{\alpha}, \rho_{\alpha} \mathbf{v}_{\alpha}, \frac{1}{2} \rho_{\alpha} v_{\alpha}^2 + \rho_{\alpha} \varepsilon_{\alpha})$ .

The proof of the statement is almost trivial: since for every  $\alpha$ , the entropy density  $\rho_{\alpha} S_{\alpha}$  is a concave function of the densities of the corresponding  $\alpha$ -fluid, then the entropy density of the mixture, being the sum of convex functions, is also a convex function of the whole densities  $\mathbf{u}$  ( $\alpha = 1, \dots, n$ ).

Let us recall that the main field components have to satisfy the relation (23)<sub>1</sub>. For the balance laws system (1) in the case of Euler fluids it reads

$$dh^0 = -d(\rho S) = \sum_{\alpha=1}^n \left\{ \tilde{\Lambda}^{\rho_{\alpha}} d\rho_{\alpha} + \tilde{\Lambda}^{\mathbf{v}_{\alpha}} d(\rho_{\alpha} \mathbf{v}_{\alpha}) + \tilde{\Lambda}^{\varepsilon_{\alpha}} d\left(\frac{1}{2} \rho_{\alpha} v_{\alpha}^2 + \rho_{\alpha} \varepsilon_{\alpha}\right) \right\} \tag{27}$$

where

$$\tilde{\mathbf{u}}' = (\tilde{\Lambda}^{\rho_{\alpha}}, \tilde{\Lambda}^{\mathbf{v}_{\alpha}}, \tilde{\Lambda}^{\varepsilon_{\alpha}}) \quad (\alpha = 1, \dots, n) \tag{28}$$

is the vector of the main field components associated to system (1).

*Statement 3*

The main field components for the mixture of Euler fluids described by system (1) have the form

$$\tilde{\Lambda}^{\rho_{\alpha}} = \frac{\mu_{\alpha} - \frac{1}{2} v_{\alpha}^2}{T_{\alpha}}, \quad \tilde{\Lambda}^{\mathbf{v}_{\alpha}} = \frac{\mathbf{v}_{\alpha}}{T_{\alpha}}, \quad \tilde{\Lambda}^{\varepsilon_{\alpha}} = -\frac{1}{T_{\alpha}} \quad (\alpha = 1, \dots, n) \tag{29}$$

where

$$\mu_{\alpha} = \varepsilon_{\alpha} - T_{\alpha} S_{\alpha} + \frac{p_{\alpha}}{\rho_{\alpha}} \tag{30}$$

are the chemical potentials of the constituents.

This statement is a consequence of the fact that system (1), for what concerns the main parts of the differential operators, is constituted by uncoupled systems of single-fluid equations. Consequently, the Gibbs relation holds for each constituent and the main field components (29) coincide with the ones obtained by Godunov [18].

The main field components of system (1) will be used for calculation of the main field of the equivalent system (7). Let us denote this main field as follows:

$$\mathbf{u}' = (\Lambda^{\rho}, \Lambda^{\mathbf{v}}, \Lambda^{\varepsilon}, \Lambda^{\rho^b}, \Lambda^{\mathbf{v}^b}, \Lambda^{\varepsilon^b}) \quad (b = 1, \dots, n - 1) \tag{31}$$

Equation (23)<sub>1</sub> written in new variables reads

$$\begin{aligned} dh^0 = -d(\rho S) = \Lambda^\rho d\rho + \Lambda^{\mathbf{v}} d(\rho \mathbf{v}) + \Lambda^\varepsilon d\left(\frac{1}{2}\rho v^2 + \rho\varepsilon\right) \\ + \sum_{b=1}^{n-1} \left\{ \Lambda^{\rho_b} d(\rho_b) + \Lambda^{\mathbf{v}_b} d(\rho_b \mathbf{v}_b) + \Lambda^{\varepsilon_b} d\left(\frac{1}{2}\rho_b v_b^2 + \rho_b \varepsilon_b\right) \right\} \end{aligned} \quad (32)$$

Expressions (27) and (32) should be equivalent and this yields the relation between the main fields  $\hat{\mathbf{u}}'$  and  $\mathbf{u}'$

*Statement 4*

The main field components for the mixture of Euler fluids described by system (7) have the form

$$\begin{aligned} \Lambda^\rho = \tilde{\Lambda}^{\rho_n}, \quad \Lambda^{\mathbf{v}} = \tilde{\Lambda}^{\mathbf{v}_n}, \quad \Lambda^\varepsilon = \tilde{\Lambda}^{\varepsilon_n} \\ \Lambda^{\rho_b} = \tilde{\Lambda}^{\rho_b} - \tilde{\Lambda}^{\rho_n}, \quad \Lambda^{\mathbf{v}_b} = \tilde{\Lambda}^{\mathbf{v}_b} - \tilde{\Lambda}^{\mathbf{v}_n}, \quad \Lambda^{\varepsilon_b} = \tilde{\Lambda}^{\varepsilon_b} - \tilde{\Lambda}^{\varepsilon_n}, \end{aligned} \quad (33)$$

for  $b = 1, \dots, n - 1$ , i.e.

$$\begin{aligned} \Lambda^\rho &= \frac{1}{T_n} \left( \mu_n - \frac{1}{2} (\mathbf{u}_n + \mathbf{v})^2 \right) \\ \Lambda^{\mathbf{v}} &= \frac{1}{T_n} (\mathbf{u}_n + \mathbf{v}) \\ \Lambda^\varepsilon &= -\frac{1}{T_n} \\ \Lambda^{\rho_b} &= \frac{1}{T_b} \left( \mu_b - \frac{1}{2} (\mathbf{u}_b + \mathbf{v})^2 \right) - \frac{1}{T_n} \left( \mu_n - \frac{1}{2} (\mathbf{u}_n + \mathbf{v})^2 \right) \\ \Lambda^{\mathbf{v}_b} &= \frac{\mathbf{u}_b}{T_b} - \frac{\mathbf{u}_n}{T_n} - \left( \frac{1}{T_b} - \frac{1}{T_n} \right) \mathbf{v} \\ \Lambda^{\varepsilon_b} &= \frac{1}{T_n} - \frac{1}{T_b} \end{aligned} \quad (34)$$

The main field permits to determine the production terms through the application of the residual inequality (23)<sub>3</sub>,  $\Sigma = \mathbf{u}' \cdot \mathbf{F} \leq 0$ . In [13], it was proved that for the main field there exists an inverse formula of (12), i.e.

$$\hat{\mathbf{u}}' = \mathbf{u}' \mathbf{X}$$

where  $\hat{\mathbf{u}}' \equiv \mathbf{u}'(\mathbf{w}, \mathbf{0})$  denotes the internal main field. In continuum theories, the internal main field coincides with the Lagrange multipliers introduced by Liu [20] in order to exploit the entropy principle.

Therefore, the residual inequality becomes

$$\Sigma = \mathbf{u}' \cdot \mathbf{F} = \hat{\mathbf{u}}' \cdot \hat{\mathbf{F}} \leq 0$$

Taking into account (15)

$$\Sigma = \hat{\mathbf{u}}' \cdot \hat{\mathbf{F}} = \sum_{b=1}^{n-1} (\hat{\Lambda}^{\rho_b} \hat{\tau}_b + \hat{\Lambda}^{\nu_b} \cdot \hat{\mathbf{m}}_b + \hat{\Lambda}^{\varepsilon_b} \hat{e}_b) \leq 0$$

or explicitly

$$\Sigma = \sum_{b=1}^{n-1} \left( \frac{\mu_b - \frac{1}{2}u_b^2}{T_b} - \frac{\mu_n - \frac{1}{2}u_n^2}{T_n} \right) \hat{\tau}_b + \left( \frac{\mathbf{u}_b}{T_b} - \frac{\mathbf{u}_n}{T_n} \right) \cdot \hat{\mathbf{m}}_b + \left( -\frac{1}{T_b} + \frac{1}{T_n} \right) \hat{e}_b \leq 0 \tag{35}$$

This inequality allows to obtain the following structure of production terms.

*Statement 5*

The internal parts of production terms (15)<sub>2</sub> are chosen in such a way that the residual inequality (35) is actually a quadratic form. In particular, in agreement with kinetic theory [21],

$$\begin{aligned} \hat{\tau}_b &= - \sum_{c=1}^{n-1} \varphi_{bc}(\mathbf{w}) \left( \frac{\mu_c - \frac{1}{2}u_c^2}{T_c} - \frac{\mu_n - \frac{1}{2}u_n^2}{T_n} \right) \\ \hat{\mathbf{m}}_b &= - \sum_{c=1}^{n-1} \psi_{bc}(\mathbf{w}) \left( \frac{\mathbf{u}_c}{T_c} - \frac{\mathbf{u}_n}{T_n} \right), \quad \hat{e}_b = - \sum_{c=1}^{n-1} \theta_{bc}(\mathbf{w}) \left( -\frac{1}{T_c} + \frac{1}{T_n} \right) \end{aligned} \tag{36}$$

where  $\varphi_{bc}(\mathbf{w})$ ,  $\psi_{bc}(\mathbf{w})$  and  $\theta_{bc}(\mathbf{w})$  are positive definite matrix functions of the objective quantities.

For processes not far from the equilibrium,  $\varphi_{bc}$ ,  $\psi_{bc}$  and  $\theta_{bc}$  depend only on the equilibrium variables  $\rho_\alpha$  and  $T$ .

### 3. PRINCIPAL SUBSYSTEMS

The use of the main field has still another advantage: the possibility of recognition of principal and equilibrium subsystems. Let us give a brief review of the results which can be found in [9].

Let us split the main field  $\mathbf{u}' \in R^N$  into two parts  $\mathbf{u}' \equiv (\mathbf{v}', \mathbf{w}')$ ,  $\mathbf{v}' \in R^M$ ,  $\mathbf{w}' \in R^{N-M}$ , ( $0 < M < N$ ) and system (25) with  $\mathbf{F} \equiv (\mathbf{f}, \mathbf{g})$ , reads

$$\partial_t \left( \frac{\partial h^0(\mathbf{v}', \mathbf{w}')}{\partial \mathbf{v}'} \right) + \partial_i \left( \frac{\partial h^i(\mathbf{v}', \mathbf{w}')}{\partial \mathbf{v}'} \right) = \mathbf{f}(\mathbf{v}', \mathbf{w}') \tag{37}$$

$$\partial_t \left( \frac{\partial h^0(\mathbf{v}', \mathbf{w}')}{\partial \mathbf{w}'} \right) + \partial_i \left( \frac{\partial h^i(\mathbf{v}', \mathbf{w}')}{\partial \mathbf{w}'} \right) = \mathbf{g}(\mathbf{v}', \mathbf{w}') \tag{38}$$

Given an assigned constant value  $\mathbf{w}'_*$  to  $\mathbf{w}'$ , we call principal subsystem of (25) the system

$$\partial_t \left( \frac{\partial h^0(\mathbf{v}', \mathbf{w}'_*)}{\partial \mathbf{v}'} \right) + \partial_i \left( \frac{\partial h^i(\mathbf{v}', \mathbf{w}'_*)}{\partial \mathbf{v}'} \right) = \mathbf{f}(\mathbf{v}', \mathbf{w}'_*) \tag{39}$$

In other words, a principal subsystem coincides with the first block of system (37) putting  $\mathbf{w}' = \mathbf{w}'_*$ .

In this case we have [9].

*Statement 6 (Subentropy law)*

The solutions of a principal subsystem (39) satisfy a supplementary subentropy law:

$$\partial_t \bar{h}^0 + \partial_i \bar{h}^i = \bar{\Sigma} \quad (40)$$

where the entropy four-vector  $\bar{h}^A(\mathbf{v}', \mathbf{w}'_*)$ , ( $A = 0, i$ ) and the entropy production  $\bar{\Sigma}$  are related to the restrictions of the entropy four-vector  $h^A(\mathbf{v}', \mathbf{w}'_*)$  and of the entropy production  $\Sigma(\mathbf{v}', \mathbf{w}'_*)$  of the full system through

$$\bar{h}^0(\mathbf{v}', \mathbf{w}'_*) = h^0(\mathbf{v}', \mathbf{w}'_*) - \mathbf{w}'_* \cdot \left( \frac{\partial h^{i0}}{\partial \mathbf{w}'^i} \right)_{\mathbf{w}' \equiv \mathbf{w}'_*}$$

$$\bar{h}^i(\mathbf{v}', \mathbf{w}'_*) = h^i(\mathbf{v}', \mathbf{w}'_*) - \mathbf{w}'_* \cdot \left( \frac{\partial h^{i0}}{\partial \mathbf{w}'^i} \right)_{\mathbf{w}' \equiv \mathbf{w}'_*}$$

$$\bar{\Sigma} = \Sigma(\mathbf{v}', \mathbf{w}'_*) - \mathbf{w}'_* \cdot \mathbf{g}(\mathbf{v}', \mathbf{w}'_*)$$

The subentropy  $\bar{h}^0$  is convex and therefore every principal subsystem is symmetric hyperbolic [9].

The requirement of the residual inequality implies that the constant  $\mathbf{w}'_*$  must satisfy  $\mathbf{w}'_* \cdot \mathbf{g}(\mathbf{v}', \mathbf{w}'_*) \equiv 0$ . This means that  $\mathbf{w}'_*$  is zero if the corresponding block equations are true balance laws, instead the constant  $\mathbf{w}'_*$  can be arbitrary if the corresponding block equations are conservation laws.

Let  $\lambda^{(k)}(\mathbf{v}', \mathbf{w}', \mathbf{n})$  and  $\bar{\lambda}^{(\bar{k})}(\mathbf{v}', \mathbf{w}'_*, \mathbf{n})$ , be the characteristic velocities of the total system (37), (38) and of subsystem (39), respectively,  $\mathbf{n} \in R^3$  being a unit vector. In general, the solutions of the subsystem are not particular solutions of the system (for  $\mathbf{w}' = \mathbf{w}'_*$ ) and the spectrum of  $\bar{\lambda}$ 's is not a part of the spectrum of  $\lambda$ 's. However, if

$$\lambda_{\max} = \max_{k=1,2,\dots,N} \lambda^{(k)}, \quad \bar{\lambda}_{\max} = \max_{\bar{k}=1,2,\dots,M} \bar{\lambda}^{(\bar{k})}$$

and similarly for the minima, one obtains the following result [9]:

*Statement 7 (Subcharacteristic conditions)*

Under the assumption that  $h^0$  is a convex function, the following sub-characteristic conditions hold for every principal subsystem [9]:

$$\begin{aligned} \lambda_{\max}(\mathbf{v}', \mathbf{w}'_*, \mathbf{n}) &\geq \bar{\lambda}_{\max}(\mathbf{v}', \mathbf{w}'_*, \mathbf{n}) \\ \lambda_{\min}(\mathbf{v}', \mathbf{w}'_*, \mathbf{n}) &\leq \bar{\lambda}_{\min}(\mathbf{v}', \mathbf{w}'_*, \mathbf{n}) \end{aligned} \quad (41)$$

$\forall \mathbf{v}' \in R^M$  and  $\forall \mathbf{n} \in R^3$ ,  $\|\mathbf{n}\| = 1$ .

An interesting case is the one in which the first  $M$  equations are conservation laws, i.e.

$$\begin{aligned} \partial_t \left( \frac{\partial h^{i0}}{\partial \mathbf{v}'^i} \right) + \partial_i \left( \frac{\partial h^{i0}}{\partial \mathbf{v}'^i} \right) &= \mathbf{0} \\ \partial_t \left( \frac{\partial h^{i0}}{\partial \mathbf{w}'^i} \right) + \partial_i \left( \frac{\partial h^{i0}}{\partial \mathbf{w}'^i} \right) &= \mathbf{g} \end{aligned}$$

In this case, the first block with  $\mathbf{w}'_* = 0$  corresponds to the principal equilibrium subsystem in which the entropy production  $-\Sigma|_E$  vanishes and attains its minimum value [9, 22].

3.1. Principal subsystems for the mixture system

Taking into account (34) and (7), we can recognize the following interesting principal subsystems:

Case 1: The ST model viewed as a principal subsystem.

Let us suppose that  $\Lambda^{\epsilon_b} = 0$  for  $b = 1, \dots, n - 1$ , then

$$T_1 = \dots = T_n = T \tag{42}$$

In other words, the ST model is obtained:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) &= 0 \\ \frac{\partial(\rho \mathbf{v})}{\partial t} + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v} - \mathbf{t}^*) &= \mathbf{0} \\ \frac{\partial(\frac{1}{2}\rho v^2 + \rho \epsilon^*)}{\partial t} + \operatorname{div} \left\{ \left( \frac{1}{2}\rho v^2 + \rho \epsilon^* \right) \mathbf{v} - \mathbf{t}^* \mathbf{v} + \mathbf{q}^* \right\} &= 0 \\ \frac{\partial \rho_b}{\partial t} + \operatorname{div}(\rho_b \mathbf{v}_b) &= \tau_b^* \\ \frac{\partial(\rho_b \mathbf{v}_b)}{\partial t} + \operatorname{div}(\rho_b \mathbf{v}_b \otimes \mathbf{v}_b + p_b^* \mathbf{I}) &= \mathbf{m}_b^* \end{aligned} \tag{43}$$

where \* indicates that the corresponding functions are evaluated for  $T_1 = T_2 = \dots = T$ .

This principal subsystem contains only the energy conservation equation for the mixture, while energy balance equations for the constituents are dropped. Thus, one may conclude that ST model naturally appears as a principal subsystem of the MT system.

Case 2: The ST model without diffusion. Let us suppose that, along with  $\Lambda^{\epsilon_b} = 0$ , we impose also  $\Lambda^{\mathbf{v}_b} = 0$  ( $b = 1, \dots, n - 1$ ):

$$\mathbf{u}_1 = \dots = \mathbf{u}_n = \mathbf{0} \tag{44}$$

We obtain in this case the following principal subsystem:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) &= 0 \\ \frac{\partial(\rho \mathbf{v})}{\partial t} + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v} + p^* \mathbf{I}) &= \mathbf{0} \\ \frac{\partial(\frac{1}{2}\rho v^2 + \rho \epsilon^*)}{\partial t} + \operatorname{div} \left\{ \left( \frac{1}{2}\rho v^2 + \rho \epsilon^* \right) \mathbf{v} + p^* \mathbf{v} \right\} &= 0 \\ \frac{\partial \rho c_b}{\partial t} + \operatorname{div}(\rho c_b \mathbf{v}) &= \tau_b \end{aligned} \tag{45}$$

where

$$p^* = \sum_{\alpha=1}^n p_\alpha(\rho_\alpha, T), \quad \rho \epsilon^* = \sum_{\alpha=1}^n \rho_\alpha \epsilon_\alpha(\rho_\alpha, T)$$

Case 3: The equilibrium subsystem. If we set

$$\Lambda^{\varepsilon_b} = \Lambda^{\mathbf{v}_b} = \Lambda^{\rho_b} = 0 \quad \forall b = 1, \dots, n-1$$

i.e.

$$T_b = T, \quad \mathbf{u}_b = 0, \quad \mu_b = \mu \quad \forall b = 1, \dots, n-1$$

we have the equilibrium Euler subsystem (a single-fluid system) with concentrations  $c_b$  being solutions of

$$\mu_1 = \mu_2 = \dots = \mu_n$$

*Remark*

If the binary mixture is without chemical reactions,  $\tau_\alpha = 0$ , the principal subsystem (45) becomes the equilibrium subsystem and the last equation (45)<sub>4</sub> yields  $c_\alpha = \text{const.}$  along the path of the corresponding particle.

### 3.2. Characteristic velocities and their upper bound in the ST model

The characteristic velocities for the MT are simple to evaluate. Since, for each constituent, they are the same as the ones of a single fluid, i.e.

$$\lambda_\alpha^{(1)} = v_{\alpha n} - c_{s\alpha}, \quad \lambda_\alpha^{(2,3,4)} = v_{\alpha n}, \quad \lambda_\alpha^{(5)} = v_{\alpha n} + c_{s\alpha}$$

where  $v_{\alpha n} = \mathbf{v}_\alpha \cdot \mathbf{n}$  are the normal component of the velocities on the wave front and  $c_{s\alpha} = \sqrt{(\partial p_\alpha / \partial \rho_\alpha)_{S_\alpha}}$  are the sound velocities. For example, for a perfect gas

$$p_\alpha = \frac{k}{m_\alpha} \rho_\alpha T_\alpha, \quad \varepsilon_\alpha = \frac{p_\alpha}{\rho_\alpha (\gamma_\alpha - 1)}$$

we have

$$c_{s\alpha} = \sqrt{\frac{k \gamma_\alpha}{m_\alpha} T_\alpha}$$

where  $k$ ,  $m_\alpha$  and  $\gamma_\alpha$  are, respectively, the Boltzmann constant, the atomic mass and the ratio of the specific heats of each constituent.

Instead, in the case of a ST system, the evaluation of the velocities is very difficult also in an equilibrium state due to the fact that the characteristic polynomial is, in general, irreducible (see e.g. [3, 23]), but thanks to the subcharacteristic property (41) of principal subsystems, we are able now to establish the following lower and upper bound for the characteristic velocities of the ST model:

$$\min_\alpha (v_{\alpha n} - c_{s\alpha}^*) \leq \lambda_{\min}^{\text{ST}}, \quad \max_\alpha (v_{\alpha n} + c_{s\alpha}^*) \geq \lambda_{\max}^{\text{ST}}$$

where now

$$c_{s\alpha}^* = \sqrt{\frac{k \gamma_\alpha}{m_\alpha} T}$$

4. QUALITATIVE ANALYSIS

System (7) is a particular case of a system of balance laws (8) and it is dissipative due to the presence of the productions that satisfy the entropy principle. Moreover, we have verified that  $h^0$  is a convex function of the densities  $\mathbf{u} \equiv \mathbf{F}^0$ . On the other hand, system (7) is of mixed type; some equations are conservation laws and the other ones are real balance laws, i.e. we are in the case in which

$$\mathbf{F}(\mathbf{u}) \equiv \begin{pmatrix} 0 \\ \mathbf{g}(\mathbf{u}) \end{pmatrix}, \quad \mathbf{g} \in \mathbb{R}^{N-M}$$

In this case, the coupling condition discovered for the first time by Shizuta–Kawashima (K-condition) [10] plays a very important role in the analysis of global existence of smooth solutions. If it is satisfied, the dissipation present in the second block of equations (balance laws) have effect also on the first block of equations (conservation laws). The K-condition reads

*In the equilibrium manifold any characteristic eigenvector is not in the null space of  $\nabla \mathbf{F}$ .* i.e.

$$\nabla \mathbf{F} \cdot \mathbf{d}^{(i)}|_E \neq 0 \quad \forall i = 1, \dots, N \tag{46}$$

where  $\mathbf{d}^{(i)}$  are the right eigenvectors of the hyperbolic system (8):

$$(\mathbf{A}_n - \lambda \mathbf{I})\mathbf{d} = 0 \tag{47}$$

$$\mathbf{A}_n = \mathbf{A}^i n_i, \quad \mathbf{A}^i = \nabla \mathbf{F}^i, \quad \mathbf{u} = \mathbf{F}^0, \quad \nabla = \frac{\partial}{\partial \mathbf{u}} \tag{48}$$

and  $E$  stands for the equilibrium state.

4.1. Global existence of smooth solutions

In fact, if the system of balance laws (8) is endowed with a convex entropy law, and it is dissipative, then the K-condition becomes a sufficient condition for the existence of global smooth solutions provided that the initial data are sufficiently smooth. Hanouzet and Natalini [24] in one-space dimension and Yong [25] in the multi-dimensional case have proved the following theorem:

*Theorem 1 (Global existence)*

Assume that system (8) is strictly dissipative and the K-condition is satisfied. Then there exists  $\delta > 0$ , such that, if  $\|\mathbf{u}(x, 0)\|_2 \leq \delta$ , there is a unique global smooth solution, which verifies

$$\mathbf{u} \in \mathcal{C}^0([0, \infty); H^2)(\mathbb{R}) \cap \mathcal{C}^1([0, \infty); H^1(\mathbb{R}).)$$

Recently, Ruggeri and Serre [26] have proved in the one-dimensional case the stability of constant states:

*Theorem 2 (Stability of a constant state)*

Under natural hypotheses of strongly convex entropy, strict dissipativeness, genuine coupling and ‘zero mass’ initial for the perturbation of the equilibrium variables, the constant solution stabilizes

$$\|\mathbf{u}(t)\|_2 = O(t^{-1/2})$$

Lou and Ruggeri [11] also observed that the weaker form of K-condition, in which we require (46) only for the right eigenvectors corresponding to genuinely nonlinear eigenvalues, is a necessary (but, in general, not sufficient) condition for the global existence of smooth solutions.

4.2. *The K-condition in the mixture theories*

For ST theory without chemical reactions it was proven [27, 28] that the K-condition is violated also for some genuinely nonlinear eigenvalues. Therefore, taking into account the results [11], in general, global smooth solutions do not exist even if the initial data are small enough. To remedy this inconvenience, in a recent paper [29] it was considered an ST mixture of Grad type in which heat conductivity and viscosity are taken into account. Instead we can prove in this section that even if we remain in the simple case of Eulerian fluids but we have an MT system, it is possible to verify that the K-condition is satisfied for all eigenvalues. For simplicity, we prove this in the case of the mixture without chemical reactions ( $\tau_\alpha = 0$ ) but of course the results are valid also in the general case.

Introducing the formal substitution on the field equation (8)

$$\partial_t \rightarrow -\lambda\delta, \quad \partial_i \rightarrow n_i\delta$$

(where  $\delta$  is a differential operator) and zero for the production terms, we obtain from (8)

$$(\mathbf{A}_n - \lambda\mathbf{I})\delta\mathbf{u} = 0$$

Therefore, from (48)  $\delta\mathbf{u} \propto \mathbf{d}$  and the K-condition becomes

$$\delta\mathbf{F}|_E = (\nabla\mathbf{F} \cdot \delta\mathbf{u})_E \propto (\nabla\mathbf{F} \cdot \mathbf{d})_E \neq 0$$

We recall that in equilibrium (see (36)<sub>2,3</sub>), we have

$$T_1 = T_2 = \dots = T_n = T$$

$$\mathbf{u}_1 = \mathbf{u}_2 = \dots = \mathbf{u}_n = \mathbf{0}$$

As a consequence, for a prescribed  $\alpha = 1, \dots, n$  we obtain:

- For  $\lambda_\alpha^{(1,5)} = v_{\alpha n} \mp c_{s\alpha}$ :

$$\delta\rho_\alpha|_E = \rho_\alpha, \quad \delta T_\alpha|_E = T(\gamma_\alpha - 1), \quad \delta\mathbf{v}_\alpha|_E = \mp c_{s\alpha}\mathbf{n}$$

$$\delta\rho_\beta|_E = \delta T_\beta|_E = \delta\mathbf{v}_\beta|_E = 0 \quad (\forall\beta = 1, \dots, n; \beta \neq \alpha)$$

and

$$\delta\rho|_E = \rho_\alpha, \quad \delta\mathbf{v}|_E = \mp c_{s\alpha} \frac{\rho_\alpha}{\rho} \mathbf{n}, \quad \delta\mathbf{u}_\beta|_E = \mp c_{s\alpha} \left(1 - \frac{\rho_\alpha}{\rho}\right) \delta_{\beta\alpha} \mathbf{n}$$

- For  $\lambda_\alpha^{(2,3,4)} = v_{\alpha n}$ :

$$\delta\rho_\alpha|_E = \rho_\alpha, \quad \delta T_\alpha|_E = -T, \quad \delta\mathbf{v}_\alpha|_E = \boldsymbol{\omega}_\alpha (\boldsymbol{\omega}_\alpha \cdot \mathbf{n} = 0)$$

$$\delta\rho_\beta|_E = \delta T_\beta|_E = \delta\mathbf{v}_\beta|_E = 0 \quad (\forall\beta = 1, \dots, n; \beta \neq \alpha)$$

and

$$\delta\rho|_E = \rho_\alpha, \quad \delta\mathbf{v}|_E = \frac{\rho_\alpha}{\rho} \boldsymbol{\omega}_\alpha, \quad \delta\mathbf{u}_\beta|_E = \left(1 - \frac{\rho_\alpha}{\rho}\right) \delta_{\beta\alpha} \boldsymbol{\omega}_\alpha$$



Therefore, taking into account (36), for a prescribed  $a = 1, \dots, n - 1$  we have:

- For  $\lambda_a^{(1,5)} = v_{a_n} \mp c_{sa}$ :

$$\delta \widehat{\mathbf{m}}_b|_E = -\psi_{ba}|_E \delta \left( \frac{\mathbf{u}_a}{T_a} \right)_E = \pm \frac{\psi_{ba}|_E}{T} c_{sa} \left( 1 - \frac{\rho_a}{\rho} \right) \mathbf{n}$$

$$\delta \widehat{e}_b|_E = \theta_{ba}|_E \delta \left( \frac{1}{T_a} \right)_E = -\frac{\theta_{ba}|_E}{T} (\gamma_a - 1)$$

- For  $\lambda_n^{(1,5)} = v_{n_n} \mp c_{sn}$  we have

$$\delta \widehat{\mathbf{m}}_b|_E = \sum_{c=1}^{n-1} \psi_{bc} \Big|_E \delta \left( \frac{\mathbf{u}_n}{T_n} \right)_E = \mp \frac{1}{T} \left( \sum_{c=1}^{n-1} \psi_{bc} \Big|_E \right) c_{sn} \left( 1 - \frac{\rho_n}{\rho} \right) \mathbf{n}$$

$$\delta \widehat{e}_b|_E = -\sum_{c=1}^{n-1} \theta_{bc} \Big|_E \delta \left( \frac{1}{T_n} \right)_E = \frac{1}{T} \left( \sum_{c=1}^{n-1} \theta_{bc} \Big|_E \right) (\gamma_n - 1)$$

Instead, for the contact wave we have

- For  $\lambda_a^{(2,3,4)} = v_{a_n}$ :

$$\delta \widehat{\mathbf{m}}_b|_E = -\frac{\psi_{ba}|_E}{T} \left( 1 - \frac{\rho_a}{\rho} \right) \boldsymbol{\omega}_a$$

$$\delta \widehat{e}_b|_E = \frac{\theta_{ba}|_E}{T}$$

- For  $\lambda_n^{(2,3,4)} = v_{n_n}$ :

$$\delta \widehat{\mathbf{m}}_b|_E = \left( \sum_{c=1}^{n-1} \psi_{bc} \Big|_E \right) \left( 1 - \frac{\rho_n}{\rho} \right) \boldsymbol{\omega}_n$$

$$\delta \widehat{e}_b|_E = -\frac{1}{T} \left( \sum_{c=1}^{n-1} \theta_{bc} \Big|_E \right)$$

Taking into account that the matrices  $\psi_{ab}$  and  $\theta_{ab}$  are positive definite, we conclude that for all the eigenvalues  $\lambda_\alpha$  ( $\forall \alpha = 1 \dots n$ ) we have  $\delta \widehat{\mathbf{m}}_b|_E \neq 0$  and  $\delta \widehat{e}_b|_E \neq 0$  at least for some  $b = 1, \dots, n - 1$  and then the K-condition  $\delta \mathbf{F}|_E \neq 0$  holds.

Therefore, if the initial data of the MT model are perturbations of equilibrium state, smooth solutions exist for all time and tends to the equilibrium constant state.

Also from this point of view the MT model provides a description more realistic than the ST model.

## 5. BINARY MIXTURE OF EULER FLUIDS

Let us turn our attention to the simplest kind of mixture: the binary (two-component) mixture of Eulerian fluids.

Recently, Ruggeri [12] introduced a change of variables in order to obtain an ST model for the mixture which resembles the structure of governing equations for a single fluid with internal structure. Defining the diffusion flux vector as

$$\mathbf{J} = \rho_1 \mathbf{u}_1 = -\rho_2 \mathbf{u}_2 = \alpha \mathbf{q}$$

where the *thermal inertia*  $\alpha$  is

$$\alpha = \frac{1}{g_1 - g_2}$$

and

$$g_\alpha = \varepsilon_\alpha + \frac{p_\alpha}{\rho_\alpha} + \frac{1}{2} u_\alpha^2 \quad (\alpha = 1, 2)$$

are the non-equilibrium enthalpies, system (7) can be written in the following form:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) &= 0 \\ \frac{\partial \rho \mathbf{v}}{\partial t} + \operatorname{div} \left( \rho \mathbf{v} \otimes \mathbf{v} + p \mathbf{I} + \frac{1}{\rho c(1-c)} \mathbf{J} \otimes \mathbf{J} \right) &= \mathbf{0} \\ \frac{\partial (\frac{1}{2} \rho v^2 + \rho \varepsilon)}{\partial t} + \operatorname{div} \left\{ \left( \frac{1}{2} \rho v^2 + \rho \varepsilon + p \right) \mathbf{v} + \left( \frac{\mathbf{v} \cdot \mathbf{J}}{\rho c(1-c)} + \frac{1}{\alpha} \right) \mathbf{J} \right\} &= 0 \\ \frac{\partial (\rho c)}{\partial t} + \operatorname{div}(\rho c \mathbf{v} + \mathbf{J}) &= \tau_1 \\ \frac{\partial (\rho c \mathbf{v} + \mathbf{J})}{\partial t} + \operatorname{div} \left\{ \rho c \mathbf{v} \otimes \mathbf{v} + \frac{1}{\rho c} \mathbf{J} \otimes \mathbf{J} + \mathbf{v} \otimes \mathbf{J} + \mathbf{J} \otimes \mathbf{v} + \nu \mathbf{I} \right\} &= \mathbf{m}_1 \\ \frac{\partial \left( \frac{1}{2} \rho c \left( \mathbf{v} + \frac{\mathbf{J}}{\rho c} \right)^2 + \rho c e \right)}{\partial t} + \operatorname{div} \left\{ \left( \frac{1}{2} \rho c \left( \mathbf{v} + \frac{\mathbf{J}}{\rho c} \right)^2 + \rho c e + \nu \right) \left( \mathbf{v} + \frac{\mathbf{J}}{\rho c} \right) \right\} &= e_1 \end{aligned} \quad (49)$$

where  $c = \rho_1/\rho$ ,  $\nu = p_1$  and  $e = \varepsilon_1$ . Equations (49)<sub>1-3</sub> are the conservation laws for the mixture, while (49)<sub>4-6</sub> are the balance laws for the first constituent  $a = 1$ . Note that (49) can be regarded as a model for a single fluid with *structure*. It is worth noticing that system (49) is equivalent to (7) for both classical and weak solutions.

For an ideal mixture, proceeding as in [23], we can define

$$p = p_1 + p_2, \quad \rho \varepsilon_I = \rho_1 \varepsilon_1 + \rho_2 \varepsilon_2$$

and write constitutive equations (thermal and caloric equations of state) as for a single fluid

$$p = \frac{k}{m(c)} \rho T(c), \quad \varepsilon_I = \frac{kT(c)}{m(c)(\gamma(c) - 1)} \tag{50}$$

provided an average atomic mass, a mixture temperature and an average ratio of specific heats are introduced:

$$\begin{aligned} \frac{1}{m(c)} &= \frac{c}{m_1} + \frac{1-c}{m_2} \\ T(c) &= c \frac{m(c)}{m_1} T_1 + (1-c) \frac{m(c)}{m_2} T_2 \\ \frac{1}{\gamma(c) - 1} &= \frac{c}{\gamma_1 - 1} \frac{m(c)}{m_1} \frac{T_1}{T(c)} + \frac{1-c}{\gamma_2 - 1} \frac{m(c)}{m_2} \frac{T_2}{T(c)} \end{aligned} \tag{51}$$

For future reference it will also be useful to express temperatures of constituents in terms of mixture temperature and the temperature difference  $\Theta = T_2 - T_1$ :

$$T_1 = T(c) - \frac{m(c)}{m_2} (1-c)\Theta, \quad T_2 = T(c) + \frac{m(c)}{m_1} c\Theta \tag{52}$$

If the MT assumption is dropped, i.e.  $T_1 = T_2 = T$  and  $\Theta = 0$ , one obtains  $T(c) \equiv T$  from (51)<sub>2</sub>, while (51)<sub>3</sub> is reduced to the average ratio of specific heats introduced in [23].

Hence, the mixture now appears as a single fluid but besides the usual fields  $(\rho, \mathbf{v}, T)$ , we have also the extended new ones  $(c, \mathbf{J}, \Theta)$ . The production term is given by (17) and (36) putting  $n = 2$ .

5.1. Main field and principal subsystems

Following the results of Section 2.4, the main field components for the binary mixture model can be easily rewritten in the new variables.

Statement 8

The main field components for the binary mixture of Euler fluids described by the system (49) have the form

$$\begin{aligned} \Lambda^\rho &= -\frac{\mu - \Delta\mu}{T - \frac{m(c)}{m_1} c\Theta} + \frac{\left(\mathbf{v} - \frac{\mathbf{J}}{\rho(1-c)}\right)^2}{2\left(T - \frac{m(c)}{m_1} c\Theta\right)} \\ \Lambda^{\mathbf{v}} &= -\frac{\mathbf{v} - \frac{\mathbf{J}}{\rho(1-c)}}{T - \frac{m(c)}{m_1} c\Theta} \\ \Lambda^\varepsilon &= \frac{1}{T - \frac{m(c)}{m_1} c\Theta} \end{aligned} \tag{53}$$

$$\Lambda^c = -\frac{\Delta\mu T - \mu_N \Theta}{T_1 T_2} + \frac{1}{T_1 T_2} \left\{ -\frac{\Theta v^2}{2} + \frac{T + \left(1 - \frac{m(c)}{m_1}\right) c \Theta}{c(1-c)} \frac{\mathbf{v} \cdot \mathbf{J}}{\rho} \right. \\ \left. + \frac{(1-2c)T - \left(1 - 2c\right) \frac{m(c)}{m_1} + c^2}{c^2(1-c)^2} \Theta \frac{J^2}{2\rho} \right\}$$

$$\Lambda^{\mathbf{J}} = \frac{1}{T_1 T_2} \left\{ \Theta \mathbf{v} - \frac{T + \left(1 - \frac{m(c)}{m_1}\right) c \Theta}{c(1-c)} \frac{\mathbf{J}}{\rho} \right\}$$

$$\Lambda^e = -\frac{\Theta}{T_1 T_2}$$

In (53)  $T_1$  and  $T_2$  are determined by Equation (52) and

$$\Delta\mu = \mu_1 - \mu_2, \quad \mu = \frac{1}{\rho}(\rho_1\mu_1 + \rho_2\mu_2), \quad \mu_N = \frac{1}{N}(N_1\mu_1 + N_2\mu_2) \quad (54)$$

where  $N_a = \rho_a/m_a$  and  $N = \rho/m(c)$  are number densities of the constituents and the mixture, respectively.

Like in Section 3, principal and equilibrium subsystems can be recognized in the case of binary mixture.

*Case 1:* If we set  $\Lambda^e = 0$ , an ST model of the binary mixture will be obtained ( $\Theta = 0$ ), see (53)<sub>6</sub>. Consequently, Equation (49)<sub>6</sub> has to be dropped and condition  $\Theta = 0$  will affect the structure of remaining equations only through constitutive equations (50)–(51). This subsystem will correspond to the system written earlier by Ruggeri [12].

*Case 2:* If it is assumed that  $\Lambda^e$  and  $\Lambda^{\mathbf{J}}$  vanish simultaneously, one will obtain a binary mixture subsystem with ST ( $\Theta = 0$ ) and without diffusion ( $\mathbf{J} = \mathbf{0}$ ), see (53)<sub>5-6</sub>. Since Equations (49)<sub>5-6</sub> have to be dropped, the model will be reduced to

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v} + p^* \mathbf{I}) = \mathbf{0} \quad (55)$$

$$\frac{\partial \left(\frac{1}{2} \rho v^2 + \rho \varepsilon^*\right)}{\partial t} + \operatorname{div} \left\{ \left(\frac{1}{2} \rho v^2 + \rho \varepsilon^* + p^*\right) \mathbf{v} \right\} = 0$$

$$\frac{\partial (\rho c)}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = \tau_1$$

where \* indicates that the corresponding functions are evaluated for  $T_1 = T_2 = T$  and  $\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{0}$ .

This model is particularly interesting due to its role in chemically reacting flows. Although it does not take into account diffusion, it is extensively used in simple analysis of detonation problems, see Fickett and Davies [30]. The only difference is that reaction rate variable is replaced by concentration variable  $c$ . On the other hand, if one analyses binary mixture without chemical reactions,  $\tau_1 = 0$ , Equation (55)<sub>4</sub> yields  $c = \text{const.}$  along the path of the particle, and a single-fluid model is obtained.

Case 3: Finally, if we set  $\Lambda^e = 0$ ,  $\Lambda^J = \mathbf{0}$  and  $\Lambda^c = 0$ , a model for a single Euler fluid will be obtained as an equilibrium subsystem of the binary mixture. Namely, Equations (49)<sub>4-6</sub> have to be dropped and (49)<sub>1-3</sub> have to be adjusted according to the consequences of our main assumptions, i.e.  $\Theta = 0$ ,  $\mathbf{J} = \mathbf{0}$  and  $c$  solution of  $\Delta\mu = \mu_1 - \mu_2 = 0$ .

5.2. Asymptotic behaviour of solutions

Considering classical solutions, the original system of balance laws (7) can be rewritten in the case of binary mixture using the material derivative in the following simple manner:

$$\begin{aligned} \frac{d\rho}{dt} + \rho \operatorname{div} \mathbf{v} &= 0 \\ \rho \frac{d\mathbf{v}}{dt} - \operatorname{div} \mathbf{t} &= \mathbf{0} \\ \rho \frac{d\varepsilon}{dt} - \mathbf{t} \cdot \nabla \mathbf{v} + \operatorname{div} \mathbf{q} &= 0 \\ \frac{d_1 \rho_1}{dt} + \rho_1 \operatorname{div} \mathbf{v}_1 &= -\varphi_{11} \left( \frac{\mu_1 - \frac{1}{2}u_1^2}{T_1} - \frac{\mu_2 - \frac{1}{2}u_2^2}{T_2} \right) \\ \rho_1 \frac{d_1 \mathbf{v}_1}{dt} + \operatorname{grad} p_1 &= \hat{\mathbf{m}}_1 = -\psi_{11} \left( \frac{\mathbf{u}_1}{T_1} - \frac{\mathbf{u}_2}{T_2} \right) \\ \rho_1 \frac{d_1 \varepsilon_1}{dt} + p_1 \operatorname{div} \mathbf{v}_1 &= \hat{\varepsilon}_1 = -\theta_{11} \left( \frac{1}{T_2} - \frac{1}{T_1} \right) \end{aligned} \tag{56}$$

where

$$\frac{d}{dt} = \partial_t + \mathbf{v} \cdot \partial_x, \quad \frac{d_1}{dt} = \partial_t + \mathbf{v}_1 \cdot \partial_x$$

and

$$\nabla \mathbf{v} \equiv \left\| \frac{\partial v_i}{\partial x_j} \right\|$$

Introducing the relaxation time:

$$\tau_T = \frac{k}{m_1} \frac{\rho_1 T_1 T_2}{\theta_{11}(\gamma_1 - 1)} \geq 0 \tag{57}$$

the last Equation (56)<sub>6</sub> assumes the form:

$$\tau_T \left( \frac{d_1 T_1}{dt} + (\gamma_1 - 1) T_1 \operatorname{div} \mathbf{v}_1 \right) = T_2 - T_1$$

Therefore, the ST model can be considered as a limit case of the MT model when relaxation time  $\tau_T$  is small enough. Taking into account that for small initial data near a constant equilibrium state the MT model has smooth global solution, we can expect that for large time the solutions of the MT model approach that of the ST model and both solutions approach together the equilibrium state. This topic is currently under investigation and will be the subject of a paper that is currently being developed.

## 6. CONCLUSIONS

It was already observed in many physical cases (see e.g. [15]) and, in particular, for the moment systems associated with the Boltzmann equation and closed by procedures of extended thermodynamics [3], that the presence of principal subsystems [9] highlights important nesting properties of the physical theories. In order to identify the principal subsystems, the main field plays a fundamental role. Also, in the case of the mixture of Eulerian fluids, we have verified that there exists this nesting structure and the ST model can be viewed as a principal subsystem of the MT model. From both the mathematical and the physical point of view, the MT appears a more satisfactory theory since it possesses good properties, such as K-condition, which guarantee the existence of global smooth solutions for small initial data. In this context, the ST appears as a rough approximation of the MT and its validity is restricted only for large time when the states are close to the equilibrium ones.

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