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Original article

Nonlinear wave propagation in binary mixtures of Euler fluids

T. Ruggeri, S. Simić*

Department of Mathematics and Research Center of Applied Mathematics (C.I.R.A.M.), University of Bologna, Via Saragozza 8, 40123 Bologna, Italy

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Abstract. In the present paper, we study the propagation of acceleration and shock waves in a binary mixture of ideal Euler fluids, assuming that the difference between the atomic masses of the constituents is negligible. We evaluate the characteristic speeds, proving that they can be separated into two groups: one is related to the case of a single Euler fluid, provided that an average ratio of specific heats is introduced; the other is new and related to the propagation speed due to diffusion. We evaluate the critical time for sound acceleration waves and compare its value to that of a single fluid. We then study shock waves, showing that three types of shock waves appear: sonic and contact shocks, which have counterparts in the single fluid case, and the diffusive shock, which is peculiar to the mixture. We discuss the admissibility of the shock waves using the Lax–Liu conditions and the entropy growth criterion. It is proved that the sonic and the characteristic shock obey the same properties as in the single fluid case, while for the diffusive shock there exists a *locally exceptional* case that is determined by a particular value of the concentration of the constituents, for which the genuine nonlinearity is lost and no shocks are admissible. For other values of the unperturbed concentration, the diffusive shock is stable in a bounded interval of admissibility.

Key words: mixture of fluids, acceleration waves, shock waves

1 Introduction

Mixtures of fluids exhibit a huge number of diverse phenomena. The problem of the formulation of an adequate mathematical model has inspired many authors and still remains unresolved. The first rational model of homogeneous mixtures of fluids was proposed by Truesdell [1] in the context of Rational Thermodynamics. The compatibility of the model with the second principle of thermodynamics was discussed by Müller in the framework of classical mechanics [2] and by Hutter and Müller for relativistic mechanics [3]. Recently, this problem was treated within the framework of Rational Extended Thermodynamics by Müller and Ruggeri [4]. As is well known, this theory removes the paradox of the infinite speed of pulse propagation and leads to an hyperbolic differential system. Here, we adopt this approach and analyze a binary mixture of ideal Euler fluids, i.e. fluids that are neither viscous nor heat conducting. In particular, we rely on the form of the differential system for binary mixtures proposed recently by Ruggeri [5]. Its basic feature is that the governing equations are in the

* On leave from the Department of Mechanics, Faculty of Technical Sciences, University of Novi Sad, Serbia
Correspondence to: T. Ruggeri (e-mail: ruggeri@ciram.unibo.it)

form of those for a single heat-conducting fluid provided that new fields are introduced: a concentration variable and the diffusion flux vector of the mixture.

The main topic of our study is the nonlinear wave propagation of acceleration waves (weak discontinuities) and shock waves. We discuss the propagation of plane (one-dimensional) waves into a region occupied by the mixture in a state of rest – without diffusion flux and with constant density, concentration, and temperature. The analysis is restricted to the case in which the difference between the molecular masses of the constituents is negligible. Although the set of mixtures obeying this property is not large, in what follows we see that this is the only case in which the qualitative analysis can be done. For different masses, only a numerical approach it is possible. In addition, from a mathematical perspective, this problem is the first step towards the treatment of more complicated models and, at the same time, an indicator of new features that are peculiar to mixtures.

What are the novel features that we can expect from this model, which is so close to the single fluid case? First, the characteristic speeds evaluated in the unperturbed field can be separated into two groups: one is related to the case of a single Euler fluid, provided that an average ratio of specific heats is introduced; the other is new and is related to the propagation speed of the second sound. A simple analysis reveals the influence of the diffusion flux vector on the characteristic velocities. A similar study was performed earlier by Leininger and Nachlinger [6].

The critical time for the acceleration waves corresponding to the fastest disturbance is obtained in the same form as in the case of a single fluid. In the past, this problem was tackled either in the purely mechanical context (see for example Leininger and Nachlinger [7] and Batra and Bedford [8]), or in the framework that includes thermomechanical phenomena and chemical reactions (like in the papers of Bowen and Chen [9,10], Bowen and Rankin [11], Romeo [12], and Torrisi and Tracinà [13]).

In the previous papers, the main goal was to obtain the Bernoulli transport equation that governs the growth and decay of the amplitude of a weak discontinuity. Our model is described by a differential system of hyperbolic type, so we were allowed to use the Bernoulli equation from the beginning thanks to the results of Boillat [14] (see also [15]), who proved that the equation governs the propagation of weak discontinuities for any hyperbolic system. Therefore the originality of our results has to be found in the qualitative analysis of the solutions of the Bernoulli equation and in particular in the comparison between the present results and those obtained for a single fluid.

Finally, in the last part of the paper we analyze the propagation of $k-$shocks. The existence of three types of shocks is proved: sonic, diffusive, and contact shocks. The first and last are the formal counterparts of the same shocks that appear in the single fluid case, and they also share the same properties concerned with shock admissibility. On the other hand, the diffusive shock, which is peculiar for the mixture, obeys the property of *local exceptionality*. This case is characterized by the existence of critical values of the field variables, in particular the concentration $c_0 = 0.5$, for which there are no admissible shocks. For other unperturbed values of the concentration, there exists a finite interval of admissibility for diffusive shocks. This remarkable and very rare property is in full accordance with the behavior of the second sound in a rigid heat conductor that has been discussed by Ruggeri et al. [16]. Although the analysis of the diffusive shock suffers from the assumption of equal masses of the constituents, we have a deep belief that this case could be extrapolated to a model of greater generality. In such a case, the present result could be used as a first approximation.

2 Binary mixtures of Euler fluids

The thermodynamic description of homogeneous mixtures is based on the metaphysical principles of Truesdell [1], which postulate for a simple mixture the same balance laws as for a single fluid.

Let us consider a binary mixture of Euler fluids, i.e. fluids that are neither viscous nor heat conducting, and do not react chemically.

In this case, following Ruggeri [5] (see also [4]), it is possible to rewrite the differential system of balance laws in the form

$$
\begin{aligned}
&\frac{\partial \rho}{\partial t}+\operatorname{div}(\rho \mathbf{v})=0, \\
&\frac{\partial(\rho c)}{\partial t}+\operatorname{div}(\rho c \mathbf{v}+\mathbf{J})=0, \\
&\frac{\partial \rho \mathbf{v}}{\partial t}+\operatorname{div}\left(\rho \mathbf{v} \otimes \mathbf{v}\ +p \mathbf{I}+\frac{1}{\rho c(1-c)}\ \mathbf{J} \otimes \mathbf{J}\right)=0, \\
&\frac{\partial(\rho c \mathbf{v}+\mathbf{J})}{\partial t}+\ \operatorname{div}\left\{\rho c \mathbf{v} \otimes \mathbf{v}+\frac{1}{\rho c}\ \mathbf{J} \otimes \mathbf{J}+(\mathbf{v} \otimes \mathbf{J}+\mathbf{J} \otimes \mathbf{v})+\nu\ \mathbf{I}\right\}=-\beta \mathbf{J}, \\
&\frac{\partial\left(\frac{1}{2} \rho v^{2}+\rho \varepsilon\right)}{\partial t}+\operatorname{div}\left\{\left(\frac{1}{2} \rho v^{2}+\rho \varepsilon+p\right) \mathbf{v}+\left(\frac{\mathbf{v} \cdot \mathbf{J}}{\rho c(1-c)}+\frac{1}{\alpha}\right) \mathbf{J}\right\}=0 .
\end{aligned}
\tag{2.1}
$$

The relations $(2.1)_1$, $(2.1)_3$, and $(2.1)_5$ represent the total conservation laws of mass, momentum, and energy, respectively, while $(2.1)_2$ and $(2.1)_4$ are, respectively, the mass balance and the momentum balance of the first constituent, and

$\rho=\rho_1+\rho_2$	is the total mass density,
$c=\frac{\rho_1}{\rho}$	is the concentration,
$\mathbf{v}=\frac{1}{\rho}\left(\rho_1\mathbf{v}_1+\rho_2\mathbf{v}_2\right),$	is the center of mass velocity,
$p=p_1+p_2,$	is the total pressure,
$\mathbf{J}=\rho_1\mathbf{u}_1=-\rho_2\mathbf{u}_2=\alpha\mathbf{q},$	is the diffusion flux vector,
$\mathbf{u}_a=\mathbf{v}_a-\mathbf{v},\ \ (a=1,2),$	are the diffusion velocities,
$\mathbf{q}_I=\sum_{a=1}^{2}\left\{\rho_a\varepsilon_a+p_a\right\}\mathbf{u}_a,$	is the intrinsic flux of internal energy,
$\mathbf{q}=\mathbf{q}_I+\frac{1}{2}\sum_{a=1}^{2}\rho_a u_a^2\mathbf{u}_a,$	is the flux of internal energy,
$\varepsilon_I=\frac{1}{\rho}\left(\rho_1\varepsilon_1+\rho_2\varepsilon_2\right),$	is the intrinsic internal energy density,
$\varepsilon=\varepsilon_I+\frac{1}{2\rho}\sum_{a=1}^{2}\rho_a u_a^2,$	is the internal energy density,

and

$$
\frac{1}{\alpha}=\left(\varepsilon_1+\frac{p_1}{\rho_1}+\frac{1}{2}u_1^2\right)-\left(\varepsilon_2+\frac{p_2}{\rho_2}+\frac{1}{2}u_2^2\right). \tag{2.2}
$$

Finally, $\nu=p_1$ and β is a production term due to the interchange of momentum between the two species.

Therefore in an extended thermodynamic model with nine fields, the binary mixture can be viewed as a single heat-conducting fluid with a variable concentration and a diffusion flux vector. It has also been noted [5] that the evolution equation $(2.1)_4$ is a natural extension of the Cattaneo equation for the heat flux.

2.1 Entropy principle and thermodynamic restrictions

The compatibility between the system (2.1) and the entropy principle is expressed in the form

$$
\frac{\partial \rho S}{\partial t}+\operatorname{div}\{\rho S\mathbf{v}+\boldsymbol{\Psi}\}\geq 0, \tag{2.3}
$$

which yields several restrictions on the constitutive equations. In particular, for non-viscous simple mixtures, we have [4]

$$\rho S = \rho_1 S_1 + \rho_2 S_2, \tag{2.4}$$

$$p_1 \equiv p_1(\rho_1, T);\ p_2 \equiv p_2(\rho_2, T);\ \varepsilon_1 \equiv \varepsilon_1(\rho_1, T);\ \varepsilon_2 \equiv \varepsilon_2(\rho_2, T), \tag{2.5}$$

such that

$$T dS_1 = d\varepsilon_1 - \frac{p_1}{\rho_1^2}\, d\rho_1, \qquad T dS_2 = d\varepsilon_2 - \frac{p_2}{\rho_2^2}\, d\rho_2, \tag{2.6}$$

$$\boldsymbol{\Psi} = \frac{\mathbf{q}_I}{T} - \frac{1}{T}\left(\rho_1 \mu_1 \mathbf{u}_1 + \rho_2 \mu_2 \mathbf{u}_2\right), \tag{2.7}$$

where $\mu_a \equiv \varepsilon_a + p_a/\rho_a - TS_a$ is the chemical potential of the constituent $a = 1, 2$. We can see that the entropy flux $\boldsymbol{\Psi}$ is not equal to $\mathbf{q}_I/T$ by definition, but it also contains an additional term that appears due to the compatibility of the balance laws and the entropy inequality. This is in full agreement with the basic ideas of extended thermodynamics.

2.2 Constitutive assumptions

In the following sections we shall analyze a binary mixture of Euler fluids that obeys the thermal equation of state of a classical ideal gas,

$$p_a = \frac{k}{m_a} \rho_a T \qquad (a = 1, 2), \tag{2.8}$$

where m_a is the atomic mass of the constituent a and k is the Boltzmann constant. The corresponding internal energy densities have the form

$$\varepsilon_a = \frac{p_a}{\rho_a(\gamma_a - 1)} \qquad (a = 1, 2), \tag{2.9}$$

where γ_a is the ratio of the specific heats of the constituent a. Therefore, the total pressure and the intrinsic internal energy density can be written in a form similar to that for the single fluid case:

$$p = \frac{k}{m(c)} \rho T, \qquad \varepsilon_I = \frac{kT}{m(c)\left(\gamma(c) - 1\right)}, \tag{2.10}$$

provided that we define the average atomic mass and the average ratio of the specific heats as

$$\frac{1}{m(c)} = \frac{c}{m_1} + \frac{1-c}{m_2}, \tag{2.11}$$

$$\frac{1}{\gamma(c) - 1} = \frac{c}{\gamma_1 - 1}\frac{m(c)}{m_1} + \frac{1-c}{\gamma_2 - 1}\frac{m(c)}{m_2}. \tag{2.12}$$

The analysis of the wave propagation based upon constitutive equations (2.10) becomes extremely complicated, even in the linear case, due to the complex interaction between the waves. It has been shown (pp. 93–94 of [4]) that the study of the linear wave propagation can be considerably simplified if the following term is neglected:

$$W = p_c + \frac{T}{\varepsilon_{I_T}} p_T \left(\frac{p_{1_T}}{\rho_1} - \frac{p_{2_T}}{\rho_2} \right).$$

Taking into account (2.8), (2.9), and (2.10), this equation is reduced to

$$W = \gamma(c) k \rho T \left(\frac{1}{m_1} - \frac{1}{m_2} \right).$$

Thus, it is obvious that W is negligible only when $m_1 \approx m_2$. If this condition is fulfilled, the average atomic mass of the mixture becomes independent of the concentration c:

$$m = m_1 \approx m_2 = \text{const}, \tag{2.13}$$

and the structure of the average ratio of the specific heats (2.12) also becomes much simpler:

$$\frac{1}{\gamma(c) - 1} = \frac{c}{\gamma_1 - 1} + \frac{1 - c}{\gamma_2 - 1}. \tag{2.14}$$

Therefore, the subsequent analysis will employ the constitutive equations

$$\begin{aligned} &p = R\rho T, \qquad \varepsilon_I = \frac{RT}{\gamma(c) - 1}, \qquad \nu = p_1 = cp, \\ &\frac{1}{\alpha} = \mathit{\Gamma} RT + \frac{(1 - 2c)J^2}{2\rho^2 c^2 (1 - c)^2}, \quad \mathit{\Gamma} = \frac{1}{\gamma_1 - 1} - \frac{1}{\gamma_2 - 1}, \end{aligned} \tag{2.15}$$

where we have introduced $R = k/m$ and $\gamma(c)$ defined by (2.14). We will also assume, without loss of generality, that $\gamma_1 < \gamma_2$. So from (2.14) we have

$$\frac{d\gamma}{dc} < 0 \quad \Leftrightarrow \quad \gamma_1 \leq \gamma(c) \leq \gamma_2 \quad \forall c \in [0, 1]. \tag{2.16}$$

3 Propagation of acceleration waves

3.1 Outline of the general theory

In this section we shall analyze the propagation of acceleration waves, i.e. weak discontinuities, in a binary mixture of ideal Euler fluids described by the evolution equations (2.1). In general, a set of balance laws can be written in the form

$$\partial_t \mathbf{F}^0(\mathbf{u}) + \partial_i \mathbf{F}^i(\mathbf{u}) = \mathbf{f}(\mathbf{u}), \tag{3.1}$$

where $\mathbf{u}(\mathbf{x}, t) \in \mathbf{R}^N$ is the unknown field vector ($\mathbf{x} = (x^i)$, $i = 1, 2, 3$, $\partial_i = \partial/\partial x^i$). By the use of the constitutive equations, this system can be rewritten in the quasi-linear form

$$\mathbf{A}^0(\mathbf{u})\partial_t \mathbf{u} + \mathbf{A}^i(\mathbf{u})\partial_i \mathbf{u} = \mathbf{f}(\mathbf{u}), \tag{3.2}$$

where $\mathbf{A}^0 = \partial \mathbf{F}^0/\partial \mathbf{u}$ and $\mathbf{A}^i = \partial \mathbf{F}^i/\partial \mathbf{u}$. Since this analysis is based on the principles of Rational Extended Thermodynamics, it is ensured that the system is *hyperbolic in the t-direction*, meaning that $\det \mathbf{A}^0 \neq 0$ and that the adjoined eigenvalue problem

$$(\mathbf{A}_n - \lambda \mathbf{A}^0)\mathbf{d} = \mathbf{0}; \qquad \mathbf{A}_n = \mathbf{A}^i n_i \tag{3.3}$$

admits $\forall \mathbf{n} \in \mathbf{R}^3$ only real eigenvalues (characteristic velocities) λ and a complete set of right eigenvectors $\mathbf{d}$. In order to study the behavior of the acceleration waves it will also be necessary to introduce the left eigenvectors $\mathbf{l}$, which are solutions of the eigenvalue problem

$$\mathbf{l}(\mathbf{A}_n - \lambda \mathbf{A}^0) = \mathbf{0}. \tag{3.4}$$

Due to the hyperbolicity, the eigenvectors can be chosen such that they satisfy the orthonormality condition $\mathbf{l}^{(i)} \mathbf{A}^0 \mathbf{d}^{(j)} = \delta^{ij}$, where δ^{ij} is the Kronecker delta, and $\mathbf{l}^{(i)}$ and $\mathbf{d}^{(j)}$ are, respectively, the left and the right eigenvectors corresponding to the eigenvalues λ^i and λ^j. Let us recall that the system (3.2) is called *genuinely nonlinear* if, for every pair of the eigenvalues and the eigenvectors,

$$\nabla_{\mathbf{u}} \lambda \cdot \mathbf{d} \neq 0. \tag{3.5}$$

On the other hand, a particular wave is called *exceptional* or *linearly degenerate* if

$$\nabla_{\mathbf{u}}\lambda \cdot \mathbf{d} \equiv 0, \tag{3.6}$$

where the operator $\nabla_{\mathbf{u}}$ denotes differentiation with respect to the field vector $\mathbf{u}$: $\nabla_{\mathbf{u}} = (\partial/\partial u^\alpha)\mathbf{e}_\alpha$, $\alpha = 1, \dots, N$, and $\mathbf{e}_\alpha$ is the standard basis of $\mathbf{R}^N$.

Due to the fact that $\mathbf{A}^0$ is non-singular, the system (3.2) for classical solutions is usually transformed into normal form:

$$\partial_t \mathbf{u} + \hat{\mathbf{A}}^i(\mathbf{u})\partial_i \mathbf{u} = \hat{\mathbf{f}}(\mathbf{u}), \tag{3.7}$$

where $\hat{\mathbf{A}}^i = (\mathbf{A}^0)^{-1}\mathbf{A}^i$ and $\hat{\mathbf{f}} = (\mathbf{A}^0)^{-1}\mathbf{f}$. In view of this change, the condition of orthonormality becomes $\mathbf{l}^{(i)}\mathbf{d}^{(j)} = \delta^{ij}$.

Let us assume that there exists a moving surface, a wave front, described by the Cartesian equation $\phi(\mathbf{x}, t) = 0$. This surface separates the space into two subspaces: ahead of the wave front we have a known unperturbed field $\mathbf{u}_0(\mathbf{x}, t)$, while behind it there is an unknown perturbed field $\mathbf{u}(\mathbf{x}, t)$. The main assumption in the analysis of acceleration waves is that both fields, $\mathbf{u}_0$ and $\mathbf{u}$, are regular solutions of the system (3.2) that are continuous across the wave front, but have discontinuous normal derivatives:

$$[\mathbf{u}] = \mathbf{0}, \qquad [\mathbf{u}_\phi] = \mathbf{\Pi} \neq \mathbf{0}. \tag{3.8}$$

Here the square bracket denotes the jump of the indicated quantity:

$$[\cdot] = (\cdot)_{\phi=0^-} - (\cdot)_{\phi=0^+}, \tag{3.9}$$

where $+$ denotes the region ahead of the wave front and $-$ the region behind it, and $\mathbf{u}_\phi = \partial\mathbf{u}/\partial\phi$.

Now, we can recall some well-known results related to the propagation of acceleration waves. First of all, if such a wave exists, the surface of propagation $\phi(\mathbf{x}, t) = 0$ is a characteristic surface. Secondly, the normal component of the propagation velocity of the wave front is equal to a characteristic velocity evaluated in unperturbed field $V = -\phi_t/|\nabla\phi| = \lambda(\mathbf{u}_0) = \lambda_0$. Thirdly, the jump vector $\mathbf{\Pi}$ is proportional to the right eigenvector $\mathbf{d}$ evaluated in the unperturbed field: $\mathbf{\Pi} = \Pi\mathbf{d}(\mathbf{u}_0)$, where Π denotes the amplitude of the jump. Finally, the amplitude Π of the jump satisfies the Bernoulli equation [14,15]:

$$\frac{d\Pi}{dt} + a(t)\Pi^2 + b(t)\Pi = 0, \tag{3.10}$$

where d/dt represents the time derivative along bicharacteristics, and $a(t)$ and $b(t)$ are known functions of the time through the unperturbed state $\mathbf{u}_0$. In the case of one space dimension we have

$$a(t) = \phi_x(\nabla\lambda \cdot \mathbf{d})_0, \tag{3.11}$$

$$b(t) = \left\{ \mathbf{d}^T \left((\nabla\mathbf{l})^T - (\nabla\mathbf{l}) \right) \cdot \frac{d\mathbf{u}}{dt} + (\nabla\lambda \cdot \mathbf{d})(\mathbf{l} \cdot \mathbf{u}_x) - \nabla(\mathbf{l} \cdot \mathbf{f}) \cdot \mathbf{d} \right\}_0, \tag{3.12}$$

$$\frac{d\phi_x}{dt} + (\nabla\lambda \cdot \mathbf{u}_x)_0 \phi_x = 0, \qquad \phi_x(0) = 1. \tag{3.13}$$

The analysis of the growth and decay of acceleration waves is based on the solution of the Bernoulli equation (3.10)

$$\Pi(t) = \frac{\Pi(0)\exp\left(-\int_0^t b(\xi)d\xi\right)}{1 + \Pi(0)\int_0^t a(\zeta)\exp\left(-\int_0^\zeta b(\xi)d\xi\right)d\zeta}. \tag{3.14}$$

From this equation one can deduce whether the acceleration wave, corresponding to the eigenvalue λ, remains bounded, vanishes asymptotically, or has its amplitude become unbounded in a finite or an infinite time interval. Following Ruggeri [15], the boundedness of the amplitude Π corresponds to the stability of the unperturbed solution $\mathbf{u}_0$ with respect to perturbations in the class of the acceleration waves. On the contrary, if $\lim_{t\to t_{crt}} |\Pi(t)| = \infty$, the unperturbed solution is unstable with respect to the same class of perturbations, and the acceleration wave evolves into a shock wave at $t = t_{crt}$.

3.2 Acceleration waves in a binary mixture

We will analyze acceleration waves propagating into an equilibrium state at rest:

$$\mathbf{v}_0 = \mathbf{0}, \qquad \mathbf{J}_0 = \mathbf{0}. \tag{3.15}$$

We will also restrict our attention to the one-dimensional case, i.e. plane waves propagating in the x-direction. In this case the field vector becomes

$$\mathbf{u} = (\rho, c, v, J, T)^T, \tag{3.16}$$

and the unperturbed state,

$$\mathbf{u}_0 = (\rho_0, c_0, 0, 0, T_0)^T, \tag{3.17}$$

identically satisfies the system of balance laws (2.1) for constant values of ρ_0, c_0, and T_0.

3.2.1 Propagation speeds and corresponding eigenvectors

After the introduction of the constitutive functions (2.15) and a lengthy calculation, which will be omitted here, the system of balance laws (2.1) can be transformed into a quasi-linear system in the normal form (3.7). By introducing the generic relative characteristic velocity $U = v - \lambda$, the characteristic polynomial $\det(\hat{\mathbf{A}} - \lambda \mathbf{I}) = 0$ becomes

$$\begin{aligned} & R\,T\,\rho^2\,(\gamma(c)-1)\,\big\{(1-2c)\,J^3 - (1-5\,c\,(1-c))\,\rho\,U\,J^2 \\ & \quad -2\,c\,(1-c)\,(1-2\,c)\,\rho^2\,U^2\,J\, - c^2\,(1-c)^2\,\rho^3\,U\,(U^2 - R\,T)\big\} \\ & \quad + \big(J^2 - 2\,(1-c)\,\rho\,U\,J\, + (1-c)^2\,\rho^2 (U^2 - R\,T)\big) \\ & \quad \times \big(J^2 - 2\,c\,\rho\,U\,J\, + c^2\,\rho^2 (U^2 - R\,T)\big)\,\left(\rho\,U + \mathit{\Gamma}\,J\,(\gamma(c)-1)\right) = 0. \end{aligned} \tag{3.18}$$

Since we are seeking solutions evaluated with respect to the unperturbed field (3.17), where $v_0 = 0$ and $J_0 = 0$, (3.18) reads

$$U_0\,\left(U_0^2 - R\,T_0\right)\,\left(U_0^2 - R\,\gamma(c_0)\,T_0\right) = 0, \tag{3.19}$$

and the following statement holds:

Statement 1 *The speeds of the acceleration waves propagating in the equilibrium unperturbed state (3.17) are equal to*

$$\lambda_0^{(1)} = -\sqrt{\gamma_0\,R\,T_0}, \qquad \lambda_0^{(3)} = 0, \qquad \lambda_0^{(5)} = \sqrt{\gamma_0\,R\,T_0}, \tag{3.20}$$

$$\lambda_0^{(2)} = -\sqrt{R\,T_0}, \qquad \lambda_0^{(4)} = \sqrt{R\,T_0}, \tag{3.21}$$

where $\gamma_0 = \gamma(c_0)$.

It is obvious from (3.20) that this set of eigenvalues has the same form as in the case of a single Euler fluid by the use of the average ratio of the specific heats (2.14). Therefore, $\lambda_0^{(1)}$ and $\lambda_0^{(5)}$ correspond to the sound velocity of a single fluid, and $\lambda_0^{(3)}$ corresponds to the so-called contact wave. These results are in substantial agreement with the fact that the evolution equations (2.1) could be treated as equations for a single heat-conducting fluid with variable concentration. The eigenvalues (3.21) correspond to the propagation speed of the *second sound* (see [4], p. 93). This is a strongly damped mode of wave propagation and it is often used to describe the propagation speed of a thermal disturbance. Recently, similar results were published by Bautin [17] on the analysis of the weak discontinuities for an inviscid heat-conducting gas based on the Navier–Stokes model.

The analysis of acceleration waves in a non-equilibrium state is considerably simplified if we neglect all terms of order $O(J^2)$ or higher in the differential system. This approximation will also simplify the structure of the internal energy density ε and thermal inertia α since $u_a^2 = J^2/\rho_a^2$. Namely, the internal energy density is reduced to its intrinsic value ε_I, which according to the constitutive equation $(2.15)_2$ reads

$$\rho\varepsilon = \rho\varepsilon_I = \frac{R\rho T}{\gamma(c) - 1}, \tag{3.22}$$

where $\gamma(c)$ is defined by (2.14). At the same time, the thermal inertia α (2.2) has the form

$$\frac{1}{\alpha} = \Gamma RT, \qquad \Gamma = \frac{1}{\gamma_1 - 1} - \frac{1}{\gamma_2 - 1}. \tag{3.23}$$

This approximation is justified by the fact that we analyze the propagation of acceleration waves in the region without diffusion flux ($J_0 = 0$) and that there is no jump in the field variables, but only in their derivatives (see (3.8)). As a consequence, all the results concerning the growth and decay of the acceleration waves will be unaffected by this approximation. We remark that in the case of shocks it is instead necessary to consider the full characteristic polynomial (3.18).

The characteristic matrix is now

$$\hat{\mathbf{A}}(\mathbf{u}) = \begin{pmatrix} v & 0 & \rho & 0 & 0 \\ 0 & v & 0 & \frac{1}{\rho} & 0 \\ R\frac{T}{\rho} & 0 & v & 0 & R \\ 0 & R\rho T & 2J & v & 0 \\ 0 & 0 & (\gamma(c)-1)T & 0 & v + \Gamma(\gamma(c)-1)\frac{J}{\rho} \end{pmatrix}, \tag{3.24}$$

$$\hat{\mathbf{f}}(\mathbf{u}) = (0, 0, 0, -\beta J, 0)^T, \tag{3.25}$$

and the characteristic equation $\det(\hat{\mathbf{A}} - \lambda\mathbf{I}) = 0$ becomes much simpler and can be factorized as

$$(U^2 - RT)\left\{(U^2 - RT)\left(U + \Gamma(\gamma(c)-1)\frac{J}{\rho}\right) - R(\gamma(c)-1)UT\right\} = 0, \tag{3.26}$$

i.e.

$$(U^2 - RT)\left(U + \Gamma(\gamma(c)-1)\frac{J}{\rho}\right) - R(\gamma(c)-1)UT = 0, \tag{3.27}$$

$$U^2 - RT = 0. \tag{3.28}$$

It is easy to verify that the speeds of propagation (3.20) come from (3.27) evaluated in the unperturbed field (3.17), while (3.21) represents the solution of (3.28).

The eigenvalues (3.20) are calculated with respect to the unperturbed field (3.17). We now discuss the influence of the diffusion flux J on the values of the propagation speeds propagating into a generic perturbed state but with small J. If we introduce the sound velocity $C = \sqrt{\gamma(c)RT}$, the dimensionless generic eigenvalue $V = U/C$, and the dimensionless diffusion flux $\widetilde{J} = J/(C\rho)$, (3.27) can be rewritten in the form

$$F(V) = V(V^2 - 1) + \Gamma(\gamma(c)-1)\left(V^2 - \frac{1}{\gamma(c)}\right)\widetilde{J} = 0. \tag{3.29}$$

For the sake of simplicity we shall restrict the analysis to the case $v_0 = 0$, that is, the case of constituents flowing with the same momentum in opposite directions. The corresponding values of the generic equilibrium characteristic speeds are $V^{(1)} = 1$, $V^{(3)} = 0$, and $V^{(5)} = -1$. Note that, comparing with the eigenvalues (3.20), a sign has changed. We can easily determine the sign of $F(V)$ in the neighborhood of $V = -1$, $V = 0$, and $V = 1$ for $\widetilde{J} \neq 0$, and deduce that the graph of the function $F(V)$ has shifted. Hence, the eigenvalues are no longer symmetric in the presence of a diffusion flux. Namely, the waves that propagate in different directions propagate with different speeds, and the contact wave does not exist anymore (see Fig. 1). The result is summarized in the following statement:

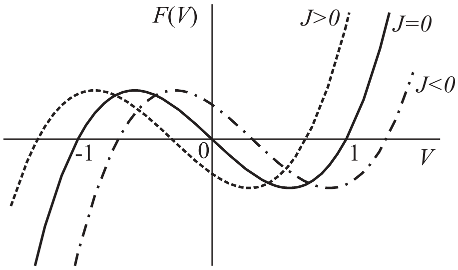

Fig. 3.1. Non-equilibrium velocities for different values of J as roots of $F(V) = 0$

Statement 2 *The characteristic velocities* $\lambda_J^{(i)}$, $i = 1, 3, 5$, *calculated with respect to the unperturbed field* $\mathbf{u}_0 = (\rho_0, c_0, 0, J_0, T_0)^T$, *have the following property in the neighborhood of* $J_0 = 0$*:*

$$\lambda_J^{(i)} \geq \lambda_0^{(i)} \quad for \quad J_0 \geq 0,$$
$$\lambda_J^{(i)} \leq \lambda_0^{(i)} \quad for \quad J_0 \leq 0,$$

where the previous equalities hold for $J_0 = 0$, *and* $\lambda_0^{(i)}$ *is determined by (3.20).*

A similar comment has already been made by Leiniger and Nachlinger [6], but our analysis gives a bit more information: by the definition of the diffusion flux vector we can deduce that the speeds of propagation $\lambda_J^{(i)}$, $i = 1, 3, 5$, are shifted in the direction of flow of constituent 1, i.e. the constituent with the lower ratio of specific heats.

At first sight, one can conclude from (3.28) that the second sound velocity depends only on the temperature of the mixture and that its form (3.21) seems to remain unchanged even when we consider wave propagation into an unperturbed region. In reality, the possibility of factorizing the characteristic equation (3.26) is a consequence of the linearization of the constitutive and evolution equations with respect to the diffusion flux J. In the general case (3.18), the characteristic equation is an irreducible polynomial of fifth degree and, through numerical analysis, it can be verified that the second sound wave is also shifted when $J \neq 0$.

The eigenvectors can now be derived from (3.3) and (3.4), and present the following form:

Statement 3 *The eigenvectors that correspond to the eigenvalues* $\lambda^{(i)}$, $i = 1, 3, 5$, *are given by*

$$\mathbf{d} = d \begin{pmatrix} 1 \\ -\frac{2JU}{\rho^2(U^2 - RT)} \\ -\frac{U}{\rho} \\ \frac{2JU^2}{\rho(U^2 - RT)} \\ \frac{T}{\rho}\left(\frac{U^2}{RT} - 1\right) \end{pmatrix}, \quad \mathbf{l} = l \begin{pmatrix} 1 \\ 0 \\ -\frac{\rho U}{RT} \\ 0 \\ \frac{\rho}{(\gamma(c)-1)T}\left(\frac{U^2}{RT} - 1\right) \end{pmatrix}^T. \tag{3.30}$$

The scalar factors d *and* l *are determined in accordance with the orthonormality condition:*

$$l = 1, \qquad d = \left\{ 1 + \frac{U^2}{RT} + \frac{1}{\gamma(c) - 1}\left(\frac{U^2}{RT} - 1\right)^2 \right\}^{-1}. \tag{3.31}$$

In the unperturbed state (3.17), they read

$$\mathbf{d_0} = d_0 \begin{pmatrix} 1 \\ 0 \\ -\frac{U_0}{\rho_0} \\ 0 \\ \frac{T_0}{\rho_0}\left(\frac{U_0^2}{RT_0} - 1\right) \end{pmatrix}, \quad \mathbf{l_0} = l_0 \begin{pmatrix} 1 \\ 0 \\ -\frac{\rho U_0}{RT_0} \\ 0 \\ \frac{\rho_0}{(\gamma_0-1)T_0}\left(\frac{U_0^2}{RT_0} - 1\right) \end{pmatrix}^T, \tag{3.32}$$

where $U_0^{(i)} = -\lambda_0^{(i)}$ *for* $i = 1, 3, 5$, *and the scalar factors have the form*

$$l_0 = 1, \qquad d_0 = \frac{1}{2\gamma_0} \quad for \quad U_0^2 = \gamma_0 R T_0, \qquad d_0 = \frac{\gamma_0 - 1}{\gamma_0} \quad for \quad U_0 = 0. \tag{3.33}$$

Statement 4 *The eigenvectors that correspond to the eigenvalues* $\lambda^{(j)}$*,* $j = 2, 4$*, have the form*

$$\mathbf{d} = d \begin{pmatrix} 0 \\ 1 \\ 0 \\ -\rho U \\ 0 \end{pmatrix}, \quad \mathbf{l} = l \begin{pmatrix} \frac{2J\left(U+\Gamma(\gamma(c)-1)\frac{J}{\rho}\right)}{\rho^2 U^2(\gamma(c)-1)} \\ 1 \\ -\frac{2J\left(U+\Gamma(\gamma(c)-1)\frac{J}{\rho}\right)}{R\rho U T(\gamma(c)-1)} \\ -\frac{1}{\rho U} \\ \frac{2J}{\rho U T(\gamma(c)-1)} \end{pmatrix}^T . \tag{3.34}$$

When they are calculated with respect to the unperturbed field (3.17), the eigenvectors read

$$\mathbf{d_0} = d_0 \begin{pmatrix} 0 \\ 1 \\ 0 \\ -\rho_0 U_0 \\ 0 \end{pmatrix}, \quad \mathbf{l_0} = l_0 \begin{pmatrix} 0 \\ 1 \\ 0 \\ -\frac{1}{\rho_0 U_0} \\ 0 \end{pmatrix}^T , \tag{3.35}$$

where $U_0^{(j)} = -\lambda_0^{(j)}$ *for* $j = 2, 4$*. The scalar factors are determined in accordance with the condition of orthonormality:*

$$d = d_0 = 1, \quad l = l_0 = \frac{1}{2}. \tag{3.36}$$

It is easy to recognize that the eigenvectors (3.32) that correspond to eigenvalues $\lambda_0^{(i)}$, $i = 1, 3, 5$, which also exist in the single fluid case, carry weak discontinuities in the field variables that describe the behavior of the mixture as a whole, i.e. the density ρ, velocity v, and temperature T. On the contrary, the eigenvectors (3.35) corresponding to $\lambda_0^{(i)}$, $i = 2, 4$, which are a peculiarity of the mixture, describe weak discontinuities in the field variables that are characteristic of the constituents, i.e. the concentration variable c and the diffusion flux J.

3.2.2 Growth of the amplitude of the sound acceleration wave

Here, we shall analyze the propagation of the fastest acceleration wave – the one that corresponds to the sound eigenvalue $\lambda_0^{(5)}$. For generic initial disturbances, this wave is the only one that propagates into the unperturbed equilibrium state.

We shall proceed with the analysis of the propagation of acceleration waves by direct application of the results (3.11)–(3.13). Since the unperturbed field (3.17) is constant, the differential equation for characteristic (3.13) reduces to $d\phi_x/dt = 0$, and in accordance with $(3.13)_2$ we have $\phi_x(t) = 1$. Since the characteristic has to satisfy the equation $\phi_t + \lambda_0^{(5)}\phi_x = 0$, we can deduce that the characteristics are described by straight lines:

$$\phi(x, t) = x - C_0 t = 0, \tag{3.37}$$

where, without loss of generality, we have set the integration constant equal to zero, and

$$C_0 = \sqrt{\gamma_0 R T_0}. \tag{3.38}$$

Using, (3.37), (3.11), and (3.12) we obtain[1]:

$$a(t) = a = \frac{1}{2\gamma_0}\frac{\gamma_0 + 1}{2\rho_0}\sqrt{\gamma_0 R T_0} = \text{const.}, \qquad b(t) = 0. \tag{3.39}$$

[1] We observe that the constitutive function β in (2.1) does not contribute to the evolution of the sound acceleration wave.

Thus, we can write the explicit solution (3.14) of the Bernoulli equation that governs the behavior of the amplitude of acceleration wave as

$$\Pi(t) = \frac{\Pi(0)}{1 + \Pi(0)at}. \tag{3.40}$$

It is obvious that $\Pi(t) \to \infty$ for an arbitrary small initial disturbance with $\Pi(0) < 0$ when $t \to t_{crt}$, which is determined by

$$t_{crt} = -\frac{1}{\Pi(0)a}. \tag{3.41}$$

In general this instant of time corresponds to the formation of a shock. The distance passed by the acceleration wave before the shock formation is determined from (3.37): $x_{crt} = C_0 t_{crt}$.

We recall that the jump of the normal derivative of the field $\mathbf{u}$ is proportional to the right eigenvector: $[\mathbf{u}_\phi] = \mathbf{\Pi} = \Pi(t)\mathbf{d}(\mathbf{u}_0)$. Hence, in the present case we have

$$[\mathbf{u}_\phi] = \begin{bmatrix} \rho_\phi \\ c_\phi \\ v_\phi \\ J_\phi \\ T_\phi \end{bmatrix} = \Pi(t)\mathbf{d}_0 = \frac{1}{2\gamma_0}\Pi(t) \begin{pmatrix} 1 \\ 0 \\ -\frac{U_0}{\rho_0} \\ 0 \\ \frac{T_0}{\rho_0}\left(\frac{U_0^2}{RT_0} - 1\right) \end{pmatrix}, \tag{3.42}$$

where $U_0 = -C_0$. Furthermore, we have that

$$[\rho_\phi](0) = \frac{1}{2\gamma_0}\Pi(0). \tag{3.43}$$

Since the existence of a positive critical time is ensured for $\Pi(0) < 0$, we can conclude that the acceleration wave evolves to a shock only for a compressive initial disturbance $[\rho_\phi](0) < 0$. Finally, we can express (3.41) in terms of the acceleration jump $G = [v_t] = [v_\phi]\phi_t$. By using

$$[v_t](0) = G_0 = -\frac{1}{2\gamma_0}\frac{C_0^2}{\rho_0}\Pi(0), \tag{3.44}$$

we obtain the expression for the critical time:

$$t_{crt}(c_0) = \frac{2C_0}{G_0(\gamma_0 + 1)}. \tag{3.45}$$

This result is formally equivalent to the expression for the critical time in a single Euler fluid. The main difference lies in the fact that the sound speed C_0 and the average ratio of the specific heats γ_0 both depend on the concentration variable c_0 in the unperturbed state. It must be emphasized that the critical time (3.45) also depends on the properties of the constituents (specific heats) through γ_0. Now we can compare this result with that of the single constituents in a rather simple manner.

Statement 5 *If we denote by* $t_{crt}^{(1)} = t_{crt}(1)$ *and* $t_{crt}^{(2)} = t_{crt}(0)$ *the critical times for fluids with ratios of the specific heats* γ_1 *and* $\gamma_2 > \gamma_1$*, respectively, the following inequality holds for the acceleration waves:*

$$t_{crt}^{(2)} = t_{crt}(0) \le t_{crt}(c_0) \le t_{crt}(1) = t_{crt}^{(1)}. \tag{3.46}$$

The proof of this statement is obtained in a straightforward manner by taking into account that, from (3.45), (3.38), (2.14), and (2.16), the function $t_{crt}(c_0)$ is a strictly increasing function of c_0 if $\gamma_2 > \gamma_1$. We can conclude that a larger concentration of the constituent with a smaller ratio of the specific heats leads to a greater critical time. In other words, its presence prolongs the so-called lifespan of the weak discontinuity. Moreover, if $\gamma_2 = \gamma_1$, the lifespan is independent of the concentration and $t_{crt}^{(1)} = t_{crt}^{(2)}$.

3.2.3 Propagation of acceleration waves in a vertical tube

We shall also analyze a problem that is slightly more involved – propagation of acceleration waves in a vertical tube. The influence of the gravity field appears in the balance equations for the momentum of the total mixture $(2.1)_3$ and for the single constituent $(2.1)_4$, as well as in the balance equation for the energy $(2.1)_5$. After some calculation we can obtain that the structure of the quasi-linear system (3.2) is changed only on the its right-hand side:

$$\mathbf{f}(\mathbf{u}) = (0, 0, -\rho g, -\beta(T)J - \rho c g, -\rho g v)^T , \tag{3.47}$$

where g is the gravity acceleration. When the equations are written in the normal form (3.7) we have

$$\hat{\mathbf{f}}(\mathbf{u}) = (0, 0, -g, -\beta(T)J, 0)^T .$$

A change will also appear in the unperturbed field $\mathbf{u}_0$ due to the influence of the gravity field:

$$\mathbf{u}_0(x) = (\tilde{\rho} e^{-gx/RT_0}, c_0, 0, 0, T_0)^T, \tag{3.48}$$

where $\tilde{\rho}$ is the density of the mixture at $x = 0$. We will give a detailed analysis only for the upward propagating acceleration wave.

In view of these changes we have to reexamine the results concerning the characteristic line and the structure of the Bernoulli equation. Fortunately, the characteristic curve remains the same as in the case of a horizontal tube (see (3.37)). After some straightforward calculations, the functions $a(t)$ and $b(t)$ given by (3.11) and (3.12) become

$$a(t) = \frac{1}{2\gamma_0} \frac{C_0(\gamma_0 + 1)}{2\tilde{\rho}} e^{gC_0 t/RT_0}, \qquad b(t) = \frac{g\gamma_0}{2C_0} = \text{const.} \tag{3.49}$$

The critical time, if it exists, is obtained as a solution of the equation

$$1 + \Pi(0) \int_0^{t_{crt}} a(\zeta) \exp\left(-\int_0^{\zeta} b(\xi) d\xi \right) d\zeta = 0. \tag{3.50}$$

and, in the present case, we obtain

$$t_{crt}(c_0) = \frac{2C_0}{g\gamma_0} \ln\left\{ 1 + \frac{g\gamma_0}{G_0(\gamma_0 + 1)} \right\}. \tag{3.51}$$

Again, this result is formally equivalent to that obtained for a single fluid [15], and we can conclude that there exists a critical time (3.51) for an arbitrary small initial amplitude of the acceleration wave $G_0 > 0$. As in the case of the horizontal tube, γ_0 and C_0 are functions of the concentration variable c_0 in the unperturbed state.

Statement 6 *If we denote by $t_{crt}^{(1)} = t_{crt}(1)$ and $t_{crt}^{(2)} = t_{crt}(0)$ the critical times for fluids with ratios of the specific heats γ_1 and $\gamma_2 > \gamma_1$, respectively, the following inequality holds for the upward propagating acceleration waves in the vertical tube:*

$$t_{crt}^{(2)} = t_{crt}(0) \leq t_{crt}(c_0) \leq t_{crt}(1) = t_{crt}^{(1)}. \tag{3.52}$$

Also, in this case the statement is true due the fact that the function $t_{crt}(c_0)$ is a strictly increasing function of c_0. Once again we can see that the constituent with the smaller ratio of the specific heats prolongs the lifespan of the acceleration wave, and that for $\gamma_1 = \gamma_2$ the critical time is independent of the concentration.

Finally, we shall give a remark about the case of a downward propagating acceleration wave, i.e. one that propagates with a speed $\lambda_0^{(1)} = -\sqrt{\gamma_0 R T_0}$. The results that one obtains are completely equivalent to the results found in the single fluid case [15]. Namely, there exists a critical, i.e. minimal, initial amplitude of the acceleration wave, which implies the formation of a shock wave in a finite time. It can be shown that the presence of the constituent with the lower ratio of the specific heats decreases the critical initial amplitude, while the lifespan of the acceleration wave remains prolonged.

4 Shock waves in binary mixtures

4.1 Rankine–Hugoniot equations and k-shocks

In this section we shall analyze another important nonlinear phenomena – shock waves. They will be treated as singular surfaces – shock fronts – where abrupt jumps of field variables occur:

$$[\mathbf{u}] \neq \mathbf{0} \quad \text{at} \quad \phi(\mathbf{x}, t) = 0. \tag{4.1}$$

In reality, thermodynamic fields are smooth in the neighborhood of a singular surface, but have steep gradients (shock structure). We will not deal with this problem in the present paper since, from a macroscopic point of view, the thickness of the shock is negligible, as in the typical Riemann problem. Moreover, we consider only the properties of single shocks and not a generic initial jump (Riemann problem), for which we know that the two states are connected by shocks and rarefaction waves.

Let us consider the system of balance laws (3.1) and let $\mathbf{n} = (n_i) = \partial_i \phi / |\nabla \phi|$ denote the unit normal of the singular surface pointing towards the unperturbed region. The shock wave is a particular weak solution of the system of balance laws (3.1) provided that the system of Rankine–Hugoniot equations across the shock front $\phi(\mathbf{x}, t) = 0$ is satisfied:

$$-s[\mathbf{F}^0(\mathbf{u})] + [\mathbf{F}^i(\mathbf{u})]n_i = 0, \tag{4.2}$$

where $s = -\partial_t \phi / |\nabla \phi|$ is the velocity of the shock wave. If the unperturbed field $\mathbf{u}_0$ in front of the shock is known and $\mathbf{n} =$ const. (which is the case for plane waves), the Rankine–Hugoniot equations represent a system of N nonlinear algebraic equations for $N + 1$ unknowns - the field variables $\mathbf{u}$ behind the front and the shock velocity s. Thus, we can choose one of these variables, or some combination of them, to characterize the *strength* of the shock generically denoted by μ.

Another interesting property of the shock waves can be observed if we introduce the mapping

$$\mathbf{\Phi}_s(\mathbf{u}) = -s\mathbf{F}^0(\mathbf{u}) + \mathbf{F}^i(\mathbf{u})n_i. \tag{4.3}$$

It is obvious that the Rankine–Hugoniot equations can now be written in the form $\mathbf{\Phi}_s(\mathbf{u}) = \mathbf{\Phi}_s(\mathbf{u}_0)$. This system always admits the trivial solution $\mathbf{u} = \mathbf{u}_0$, which corresponds to the absence of a shock. For a nontrivial solution $\mathbf{u} \neq \mathbf{u}_0$, the mapping (4.3) must be locally non-invertible. Since the condition of local non-invertibility requires

$$\det\left(\frac{\partial \mathbf{\Phi}_s}{\partial \mathbf{u}}\right) = \det\left(\mathbf{A}^i n_i - s\mathbf{A}^0\right) = 0, \tag{4.4}$$

it follows that it is fulfilled for $s = \lambda(\mathbf{u})$, i.e. for a shock velocity equal to a characteristic speed.

We will analyze only *k-shocks*, i.e. solutions of the Rankine–Hugoniot equations that bifurcate from the trivial one $\mathbf{u} = \mathbf{u}_0$, i.e.

$$\lim_{\mu \to \mu_0} \mathbf{u}(\mu) = \mathbf{u}(\mu_0) = \mathbf{u}_0, \tag{4.5}$$

$$\lim_{\mu \to \mu_0} s(\mu) = s(\mu_0) = \lambda(\mathbf{u}_0),$$

where μ_0 is the value of the shock parameter corresponding to the null shock.

Moreover, in the case of *k*-shocks the jump vector of the field variables has the property

$$[\mathbf{u}] \propto \mathbf{d}_0 \quad \text{when the shock become weak}, \tag{4.6}$$

where $\mathbf{d}_0$ represents the right eigenvector evaluated in the unperturbed field.

4.2 Shock admissibility

As is well known, not all the solutions of the Rankine–Hugoniot equations represent physically admissible shocks. We recall briefly in this section the admissibility condition for the shocks when the system of balance laws also satisfies an entropy law:

$$\partial_t h^0 + \partial_i h^i = \Sigma \leq 0, \tag{4.7}$$

where h^0 is a convex function of the densities $\mathbf{u} \equiv \mathbf{F}^0$. We have three different cases depending on the property of the characteristic eigenvalue λ corresponding to the bifurcation point of a given k-shock.

Case 1: Genuine nonlinearity. This case is defined by the condition

$$\nabla_{\mathbf{u}} \lambda \cdot \mathbf{d} \neq 0 \quad \text{for all} \quad \mathbf{u}. \tag{4.8}$$

In this case the shock is admissible if and only if

$$\lambda_0 < s < \lambda \qquad \text{(Lax condition)}. \tag{4.9}$$

This condition is equivalent (for k-shocks) to the entropy growth condition associated with (4.7) (see [18]):

$$\eta = -s[h^0] + [h^i]n_i > 0 \qquad \text{(entropy growth)}. \tag{4.10}$$

Case 2: Linearly degenerate case (exceptional wave). In this case the following condition holds:

$$\nabla_{\mathbf{u}} \lambda \cdot \mathbf{d} \equiv 0 \quad \text{for all} \quad \mathbf{u}. \tag{4.11}$$

This corresponds to the so-called characteristic shock:

$$s = \lambda_0 = \lambda. \tag{4.12}$$

The characteristic shock is determined by m parameters ξ_I, $I = 1, \ldots, m$, where m is the algebraic multiplicity of the characteristic eigenvalue λ. In this case we have [19]

$$\eta \equiv 0. \tag{4.13}$$

Case 3: Local exceptionality. This case is defined by the following relation:

$$\nabla_{\mathbf{u}} \lambda \cdot \mathbf{d} = 0 \quad \text{for some} \quad \tilde{\mathbf{u}}. \tag{4.14}$$

In this situation the Lax condition (4.9), as well as the entropy growth condition (4.10), are not sufficient for the admissibility. The appropriate requirement is expressed by the so-called Liu condition [20,21], which states that a shock $\mathbf{u}(\mathbf{u}_0, \mu)$ is admissible if and only if

$$s(\mathbf{u}_0, \bar{\mu}) \leq s(\mathbf{u}_0, \mu) \qquad \text{for all} \qquad \mu_0 \leq \bar{\mu} \leq \mu. \tag{4.15}$$

The condition (4.15) implies the generalized Lax condition

$$\lambda_0 \leq s \leq \lambda, \tag{4.16}$$

while the entropy growth condition is not sufficient to guarantee (4.15). It is necessary to add to the entropy growth a new principle of *superposition of shocks* [22]. Of course, the admissibility conditions of Case 1 are contained in the ones of Case 3. We call $\mathcal{C}$ the set of all *critical values* $\tilde{\mathbf{u}}$ for which (4.14) is satisfied. In many physical situations $\mathcal{C}$ is a manifold in the space of state variables – the *critical manifold*.

A particular and important case is when $\mathbf{u}_0 \in \mathcal{C}$, which implies $\nabla_{\mathbf{u}} \lambda \cdot \mathbf{d}|_{\mathbf{u}_0} = 0$. In this case $\lambda(\mathbf{u}_0, \mu)$ has a stationary point at $\mu = \mu_0$:

$$\dot{\lambda}_0 = \left.\frac{\partial \lambda}{\partial \mu}\right|_{\mu=\mu_0} \propto \nabla_{\mathbf{u}} \lambda \cdot \mathbf{d}|_{\mathbf{u}_0} = 0. \tag{4.17}$$

The stationary point of $\lambda(\mathbf{u}_0, \mu)$ is also a stationary point of $s(\mathbf{u}_0, \mu)$ for weak shocks. In fact in the neighborhood of the null shock we have $2\dot{s} = \dot{\lambda}_0$ [18]. Moreover if $\lambda(\mathbf{u}_0, \mu)$ has a maximum for $\mu = \mu_0$, then $s(\mathbf{u}_0, \mu)$ also reaches the maximum for the same value of the shock parameter [16]. Under these circumstances, condition (4.15) is violated for any value of $\bar{\mu}$ in the neighborhood of μ_0. This implies that there are no admissible weak shocks.

4.3 k-shocks in binary mixtures

In contrast to the case of acceleration waves, for which the approximation concerning the linearity of the diffusion flux J does not affect the general conclusions, in the study of shock waves we have to give the analysis with the complete set of balance equations (2.1). In the one-dimensional case they read

$$\begin{aligned}
&\partial_t\rho + \partial_x(\rho v) = 0,\\
&\partial_t(\rho c) + \partial_x(\rho c v + J) = 0,\\
&\partial_t(\rho v) + \partial_x\left(\rho v^2 + p + \frac{J^2}{\rho c(1-c)}\right) = 0,\\
&\partial_t(\rho c v + J) + \partial_x\left(\rho c v^2 + 2vJ + \frac{J^2}{\rho c} + \nu\right) = -\beta(T)J,\\
&\partial_t\left(\frac{1}{2}\rho v^2 + \rho\varepsilon\right) + \partial_x\left(\left(\frac{1}{2}\rho v^2 + \rho\varepsilon + p\right)v + \frac{J^2}{\rho c(1-c)}v + \frac{1}{\alpha}J\right) = 0.
\end{aligned} \tag{4.18}$$

These equations will be adjoined with the general constitutive equations (2.15). The internal energy ε and the thermal inertia α become

$$\rho\varepsilon = \frac{p}{\gamma(c)-1} + \frac{J^2}{2\rho c(1-c)}, \quad \frac{1}{\alpha} = \Gamma\frac{p}{\rho} + \frac{(1-2c)J^2}{2\rho^2 c^2(1-c)^2}, \tag{4.19}$$

where $\gamma(c)$ is defined by (2.14).

Let us introduce the relative velocity u of the shock wave with respect to a fluid mixture:

$$u = s - v. \tag{4.20}$$

Using this quantity, after some calculations, we can write the Rankine–Hugoniot equations for the mixture in the form

$$\begin{aligned}
&[\rho u] = 0,\\
&[\rho c u - J] = 0,\\
&\left[\rho u^2 + p + \frac{J^2}{\rho c(1-c)}\right] = 0,\\
&\left[\rho c u^2 - 2Ju + \frac{J^2}{\rho c} + \nu\right] = 0,\\
&\left[\left(\frac{1}{2}\rho u^2 + \rho\varepsilon\right)u + pu + \frac{J^2}{\rho c(1-c)}u - \frac{1}{\alpha}J\right] = 0.
\end{aligned} \tag{4.21}$$

From (4.20) the entropy production η across the shock is reduced to

$$\eta = [\rho u S] - [\Psi], \tag{4.22}$$

where the entropy S and the entropy flux Ψ are given by

$$S = \frac{R}{\gamma(c)-1}\ln\left(\frac{p}{\rho^{\gamma(c)}}\right) - R\left\{c\ln c + (1-c)\ln(1-c)\right\}, \tag{4.23}$$

$$\Psi = RJ\left\{\Gamma\ln\left(\frac{p}{\rho}\right) - \ln\left(\frac{c}{1-c}\right)\right\}. \tag{4.24}$$

Let us assume that the unperturbed field $\mathbf{u}_0$ is described by

$$\mathbf{u}_0 = (\rho_0, c_0, v_0, 0, T_0)^T, \tag{4.25}$$

where ρ_0, c_0, v_0, and T_0 are constants.

In order to discuss the exact solutions of the Rankine–Hugoniot equations (4.21), we shall introduce the dimensionless quantities

$$
\begin{aligned}
r &= \frac{\rho}{\rho_0} \text{ (density)},\\
w &= \frac{u}{u_0} \text{ (velocity)},\\
\pi &= \frac{p}{p_0} \text{ (pressure)},\\
\theta &= \frac{T}{T_0} \text{ (temperature)},
\end{aligned}
$$

and

$$M_0 = \frac{u_0}{C_{0S}} \text{ (sonic Mach number)}, \tag{4.26}$$

$$\omega_0 = \frac{u_0}{C_{0D}} \text{ (diffusive Mach number)}, \tag{4.27}$$

where $C_{0S}^2 = \gamma_0 R T_0$, $C_{0D}^2 = R T_0$ and $\gamma_0 = \gamma(c_0)$. Here, we will discuss only shocks with positive velocity, since the analysis of k-shocks with negative values of the propagation speed is completely equivalent because of the symmetry of the characteristic speeds in the unperturbed state.

We can easily solve $(4.21)_1$ and $(4.21)_2$ to obtain

$$r = \frac{1}{w}, \qquad J = \rho_0 u_0 (c - c_0). \tag{4.28}$$

By introducing these results into $(4.21)_3$ we get the relation between the velocity and the pressure

$$\pi = 1 + \frac{\rho_0 u_0^2}{p_0} \left\{ 1 - w \left(1 + \frac{(c - c_0)^2}{c(1-c)} \right) \right\}. \tag{4.29}$$

By using (4.29), $(4.21)_4$ yields the relation between the concentration c and the velocity w

$$(c - c_0) \left\{ 1 + \frac{\rho_0 u_0^2}{p_0} \left(1 - w \left(2 - \frac{c - c_0}{c} + \frac{c - c_0}{1 - c} \right) \right) \right\} = 0. \tag{4.30}$$

Obviously, this equation has two independent solutions that lead to two distinct cases, which will be treated separately.

Case 1: Sonic shock. The first solution of (4.30) is

$$c = c_0, \tag{4.31}$$

i.e. there is no jump of the concentration, and from $(4.28)_2$ we also conclude that there is no diffusion flux ($J \equiv J_0 = 0$) in the perturbed state. At this stage we already get a hint that this case corresponds to the propagation of a *sonic* k-shock bifurcating from the eigenvalue $\lambda_0^{(5)}$ corresponding to the speed of sound. The proof of this statement will be given below.

By introducing (4.31) and $J \equiv 0$ into (4.29) we obtain a new relation between pressure and velocity:

$$\pi = 1 + \frac{\rho_0 u_0^2}{p_0} (1 - w), \tag{4.32}$$

and $(4.21)_5$ can be transformed into

$$\frac{1}{2} \frac{\rho_0 u_0^2}{p_0} \left(w^2 - 1 \right) + \frac{\gamma_0}{\gamma_0 - 1} (\pi w - 1) = 0. \tag{4.33}$$

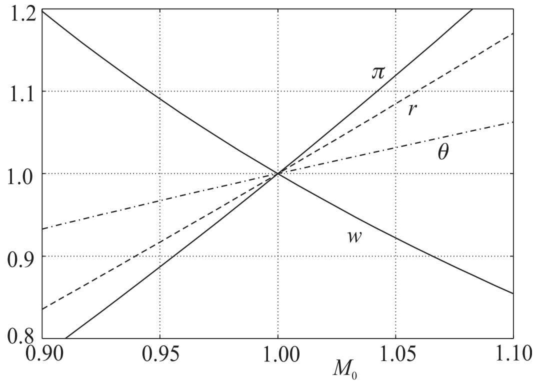

Fig. 4.1. Jumps of field variables in the sonic shock

Eliminating the velocity w from the last two equations we arrive at the following relation between the jump of the pressure and the relative shock velocity $u_0 = s - v_0$:

$$(\pi - 1)\left\{\frac{1}{\gamma_0 - 1} - \frac{1}{2}\frac{p_0}{\rho_0 u_0^2}\left(1 + \frac{\gamma_0 + 1}{\gamma_0 - 1}\,\pi\right)\right\} = 0. \tag{4.34}$$

Apart from the trivial solution $\pi = 1$, which corresponds to the null shock solution, we have another solution corresponding to a genuine jump of the pressure:

$$\pi = \frac{2}{\gamma_0 + 1}\,\frac{\rho_0 u_0^2}{p_0} - \frac{\gamma_0 - 1}{\gamma_0 + 1}. \tag{4.35}$$

It is a simple issue to verify that $\pi \to 1$ when $u_0 \to C_{0S} = \lambda_0^{(5)}$, i.e. in the case of a sonic shock, as was already noted. On the other hand, eliminating the relative shock velocity u_0 from (4.32) and (4.33) and using $(4.28)_1$ one can obtain the well-known equation for the adiabatic-shock

$$\pi = \frac{(\gamma_0 + 1)\,r - (\gamma_0 - 1)}{(\gamma_0 + 1) - (\gamma_0 - 1)\,r}. \tag{4.36}$$

We proceed in the traditional way and present the results in a compact form using the sonic Mach number (4.26) as the shock parameter.

Statement 7 *For the sonic shock we have the following nontrivial solution of the Rankine–Hugoniot equations:*

$$\pi = \left(1 + \mu_0^2\right) M_0^2 - \mu_0^2, \tag{4.37}$$

$$w = \frac{1}{r} = \frac{1}{M_0^2}\left\{1 - \mu_0^2\left(1 - M_0^2\right)\right\}, \tag{4.38}$$

$$\theta = \frac{\pi}{r} = \pi w = \frac{1}{M_0^2}\left\{\left(1 + \mu_0^2\right) M_0^2 - \mu_0^2\right\}\left\{1 - \mu_0^2\left(1 - M_0^2\right)\right\}, \tag{4.39}$$

$$c \equiv c_0, \qquad J \equiv 0, \tag{4.40}$$

where $\mu^2(c) = \frac{\gamma(c)-1}{\gamma(c)+1}$ *and* $\mu_0 = \mu(c_0)$.

Therefore, in the case of the sonic shock, one obtains results that are *formally* the same as in the case of a single fluid. Nevertheless, they still contain information about the unperturbed state of the mixture through the average ratio of the specific heats $\gamma_0 = \gamma(c_0)$, i.e. $\mu_0 = \mu(c_0)$. This shock does not produce a jump of the concentration or diffusion flux – variables that characterize the behavior of the constituents. Figure 2 shows the jumps of the field variables in the sonic shock as functions of the sonic Mach number M_0 for $\gamma_1 = 1.35$, $\gamma_2 = 1.40$, and $c_0 = 0.3$.

It can be proved that the wave that propagates at the sound speed satisfies the condition of genuine nonlinearity (4.8) and, therefore, in order to pick up physically admissible shocks we exploit the Lax condition (4.9), as well as the entropy growth condition (4.10). The Lax condition $\lambda_0 < s < \lambda$ can be rewritten in dimensionless form

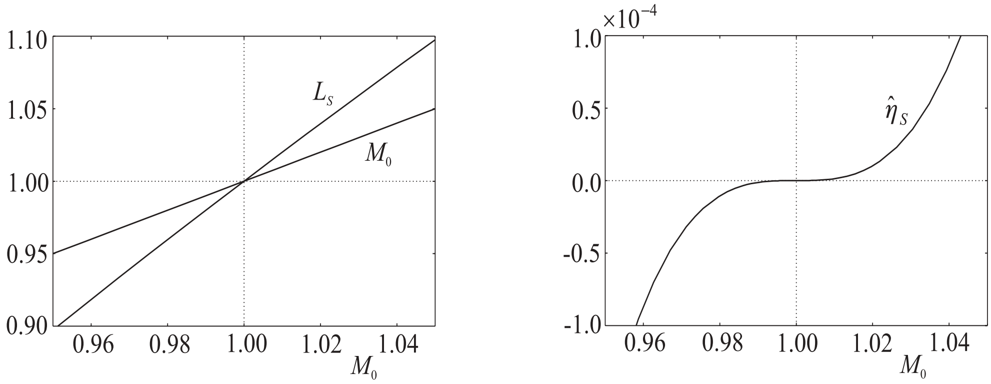

Fig. 4.2. Lax condition and entropy growth in the sonic shock

as

$$1 < M_0 < L_S = \Lambda_S - M_0\,(w-1), \tag{4.41}$$

where $\Lambda_S = (\lambda - v)/C_{0S}$. At the same time the dimensionless entropy growth condition reads

$$\hat{\eta}_S = \frac{\eta}{R\,\rho_0\,C_{0S}} = \frac{M_0}{\gamma_0 - 1}\ln\left(\frac{\pi}{r^{\gamma_0}}\right) \geq 0. \tag{4.42}$$

It is easy to verify (see Fig. 3), in accordance with the theory, that for weak shocks bifurcating from $M_0 = 1$ these conditions lead to the same conclusion.

Statement 8 *In the case of a sonic shock, the physically admissible shocks appear for $M_0 > 1$. In this case*

- *pressure jumps up,*
- *density jumps up,*
- *velocity jumps down, and*
- *temperature jumps up.*

Thus, we have behavior equivalent to that in the single fluid: an admissible shock is supersonic with respect to the unperturbed field and subsonic with respect to the perturbed one.

Case 2: Diffusive shock. Starting from the second solution of (4.30) we can express the jump of the velocity w in terms of the concentration variable c and the diffusive Mach number $\omega_0 = u_0/C_{0D}$ as

$$w = \frac{1}{\alpha(c)}\,\frac{1+\omega_0^2}{\omega_0^2}. \tag{4.43}$$

Introducing this result into (4.29) one obtains the corresponding expression for the jump of pressure π,

$$\pi = \left(1 - \frac{\beta(c)}{\alpha(c)}\right)(1+\omega_0^2). \tag{4.44}$$

In (4.43)–(4.44) we used

$$\alpha(c) = 2 - \frac{c-c_0}{c} + \frac{c-c_0}{1-c}, \quad \beta(c) = 1 + \frac{(c-c_0)^2}{c\,(1-c)}. \tag{4.45}$$

By a straightforward calculation one can verify that (4.43)–(4.44) describe a k-shock bifurcating from the critical point $\omega_0 = 1$ corresponding to the propagation speed of $\lambda_0^{(4)} = \sqrt{RT_0}$. We call this shock a *diffusive shock* since it predominantly carries jumps of concentration c and diffusion flux J. The appearance of this kind of k-shock is a peculiarity of the mixture.

Under these circumstances, $(4.21)_5$ leads to a fundamental equation for the determination of the diffusive shock waves:

$$\begin{aligned}&\frac{1}{2}\,\omega_0^2\,(w^2-1)+\frac{\gamma_0}{\gamma_0-1}\,(\pi\,w-1)\\&+\frac{3}{2}\,\omega_0^2\,w^2\,\frac{(c-c_0)^2}{c\,(1-c)}-\omega_0^2\,w^2\,\frac{(1-2c)(c-c_0)^3}{2c^2(1-c)^2}=0.\end{aligned} \tag{4.46}$$

By introducing (4.43) and (4.44) into (4.46), the fundamental equation becomes a nonlinear one relating c and ω_0:

$$a_0(c)\omega_0^4+a_1(c)\omega_0^2+a_2(c)=0. \tag{4.47}$$

Here, we have

$$\begin{aligned}a_0(c)=&-\frac{1}{2}+\frac{1}{2\,\alpha^2(c)}+\frac{3\,(c-c_0)^2}{2\,c\,(1-c)\,\alpha^2(c)}\\&-\frac{(1-2\,c)\,(c-c_0)^3}{2\,c^2\,(1-c)^2\,\alpha^2(c)}+\frac{\gamma_0}{\gamma_0-1}\,\frac{1}{\alpha(c)}\,\left(1-\frac{\beta(c)}{\alpha(c)}\right),\\a_1(c)=&-\frac{\gamma_0}{\gamma_0-1}+\frac{1}{\alpha^2(c)}+\frac{3\,(c-c_0)^2}{c\,(1-c)\,\alpha^2(c)}\\&-\frac{(1-2\,c)\,(c-c_0)^3}{c^2\,(1-c)^2\,\alpha^2(c)}+\frac{2\,\gamma_0}{\gamma_0-1}\,\frac{1}{\alpha(c)}\,\left(1-\frac{\beta(c)}{\alpha(c)}\right),\\a_2(c)=&\,\frac{1}{2\,\alpha^2(c)}+\frac{3\,(c-c_0)^2}{2\,c\,(1-c)\,\alpha^2(c)}\\&-\frac{(1-2\,c)\,(c-c_0)^3}{2\,c^2\,(1-c)^2\,\alpha^2(c)}+\frac{\gamma_0}{\gamma_0-1}\,\frac{1}{\alpha(c)}\,\left(1-\frac{\beta(c)}{\alpha(c)}\right).\end{aligned}$$

We can easily deduce that one branch of the solution, determined by

$$\omega_0^2(c)=-\frac{a_1(c)}{2a_0(c)}-\sqrt{\left(\frac{a_1(c)}{2a_0(c)}\right)^2-\frac{a_2(c)}{a_0(c)}}, \tag{4.48}$$

bifurcates from $\omega_0=1$, while another branch does not pass through the critical point and consequently does not correspond to a diffusive k-shock . For this kind of shock, it is convenient to choose as the shock-free parameter the perturbed concentration variable c. We summarize the results related to (4.48), (4.43), (4.44), and (4.28) as follows:

Statement 9 *For the diffusive shock we have the following non-trivial solution of the Rankine–Hugoniot equations:*

$$\begin{aligned}&\omega_0=\left\{-\frac{a_1(c)}{2a_0(c)}-\sqrt{\left(\frac{a_1(c)}{2a_0(c)}\right)^2-\frac{a_2(c)}{a_0(c)}}\right\}^{1/2},\\&w=\frac{1}{r}=\frac{1}{\alpha(c)}\,\frac{1+\omega_0^2(c)}{\omega_0^2(c)};\\&\pi=\left(1-\frac{\beta(c)}{\alpha(c)}\right)\,(1+\omega_0^2(c)),\\&\theta=\frac{\pi}{r}=\pi\,w=\frac{1}{\alpha(c)}\,\left(1-\frac{\beta(c)}{\alpha(c)}\right)\,\frac{\left(1+\omega_0^2(c)\right)^2}{\omega_0^2(c)},\\&J=\rho_0\,C_{0D}\,\omega_0(c)\,(c-c_0).\end{aligned}$$

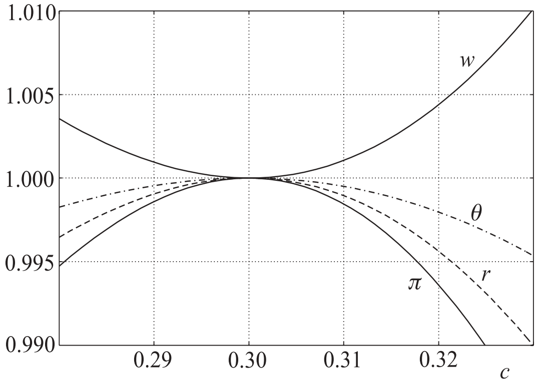

Fig. 4.3. Jumps of field variables in a diffusive shock for $c_0 = 0.3$

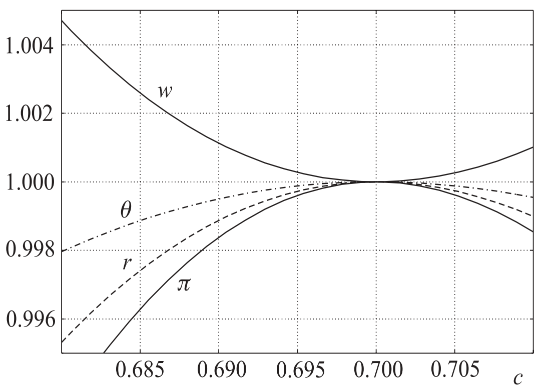

Fig. 4.4. Jumps of field variables in a diffusive shock for $c_0 = 0.7$

Let us recall that the jumps of the field variables for a weak shock are proportional to the right eigenvector evaluated in the unperturbed field (see (4.6)). Due to the structure of the right eigenvector (3.35), one can deduce that the jump of the diffusion flux J is of order $O(c-c_0)$, while the jumps of the density r, velocity w, pressure π, and temperature θ are of order $O\left((c-c_0)^2\right)$, and are thus independent of the sign of $c-c_0$ in the neighborhood of c_0. Figures 4–6 show the jumps of the field variables as functions of the perturbed concentration c for $\gamma_1 = 1.35$ and $\gamma_2 = 1.40$.

This result motivates a more careful discussion about the influence of the unperturbed value c_0 on the jumps of the field variables. Namely, a simple numerical study reveals that for different values of c_0, the sign of $(d\omega_0(c)/dc)_{c=c0}$ and, consequently, its monotonicity are different. Therefore there must exist a value c_0^* such that the condition of local exceptionality (4.17) is satisfied. In fact, after some calculations based on the characteristic equation (3.18) we get

$$\nabla_{\mathbf{u}}\lambda^{(4)} \cdot \mathbf{d}^{(4)}\Big|_{\mathbf{u}_0} = \frac{1-2\,c_0}{c_0\,(1-c_0)}\,\sqrt{R\,T_0},$$

which leads to the following conclusion:

Statement 10 *The eigenvalue $\lambda^{(4)}$ obeys the condition of local exceptionality (4.14) with respect to the unperturbed field $\mathbf{u}_0 = (\rho_0, c_0, v_0, 0, T_0)$ for*

$$c_0^* = 0.5 \tag{4.49}$$

and arbitrary unperturbed values of the state variables ρ_0, v_0, and T_0.

Therefore the admissibility analysis for the diffusive shock has to be performed separately for the cases $c_0 < c_0^*$, $c_0 = c_0^*$, and $c_0 > c_0^*$, since we lose the genuine nonlinearity for $c_0 = c_0^*$. In order to employ the Liu condition for the study of the admissible shocks we transform the characteristic equation (3.18) into the

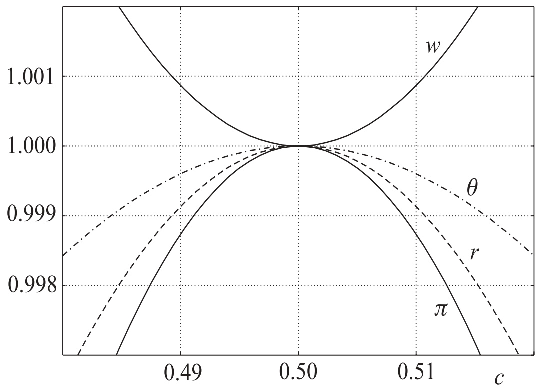

Fig. 4.5. Jumps of field variables in a diffusive shock for $c_0 = 0.5$

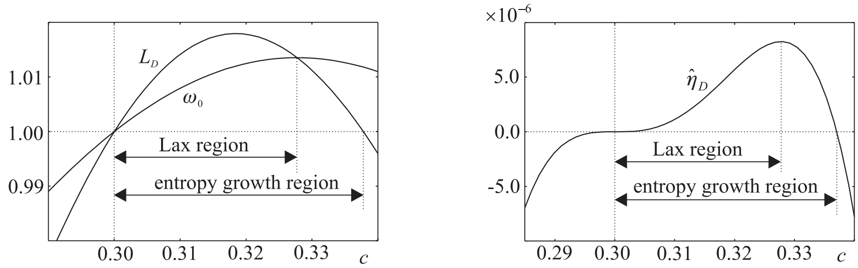

Fig. 4.6. Lax condition and entropy growth in a diffusive shock for $c_0 = 0.3$

dimensionless form

$$
\begin{aligned}
&\pi\, r\,(\gamma(c)-1)\,\left(c^2\,(1-c)^2\,r^3\,\Lambda_D\,\left(\Lambda_D^2-\frac{\pi}{r}\right)\right.\\
&\quad -2\,c\,(1-c)\,(1-2\,c)\,(c-c_0)\,\omega_0\,r^2\,\Lambda_D^2\\
&\quad \left.+(1-5\,c\,(1-c))\,(c-c_0)^2\,\omega_0^2\,r\,\Lambda_D+(1-2\,c)\,(c-c0)^3\,\omega_0^3\right)\\
&\quad +\left((1-c)^2\,r^2\,\left(\Lambda_D^2-\frac{\pi}{r}\right)+2\,(1-c)\,(c-c_0)\,\omega_0\,r\,\Lambda_D+(c-c_0)^2\,\omega_0^2\right)\\
&\quad\ \times\left(c^2\,r^2\,\left(\Lambda_D^2-\frac{\pi}{r}\right)-2\,c\,(c-c_0)\,\omega_0\,r\,\Lambda_D+(c-c_0)^2\,\omega_0^2\right)\\
&\quad\ \times\left((\gamma(c)-1)\,(c-c_0)\,\Gamma\,\omega_0-r\,\Lambda_D\right)=0,
\end{aligned}
\tag{4.50}
$$

where $\Lambda_D = (\lambda - v)/C_{0D}$. The corresponding dimensionless generalized Lax condition reads

$$
1 \leq \omega_0 \leq L_D = \Lambda_D - \omega_0\,(w-1). \tag{4.51}
$$

We also evaluate the entropy growth condition, which, in the case of a diffusive shock, is

$$
\hat{\eta}_D = \frac{\eta}{R\,\rho\,C_{0D}} = \omega_0\,\left\{\frac{1}{\gamma_0-1}\,\ln\left(\frac{\pi}{r^{\gamma_0}}\right)-(f(c)-f(c_0))+(c-c_0)\,\ln\left(\frac{c}{1-c}\right)\right\}, \tag{4.52}
$$

where

$$
f(c) = c\,\ln c + (1-c)\,\ln(1-c).
$$

Since the admissibility analysis for the diffusive shocks is much more involved than for the sonic shocks, we will follow the results of the numerical solution of (4.50) in the neighborhood of $\omega_0 = 1$ (see Figs. 7–9), which appears to be sufficient for our conclusions . The values of the specific heats are kept the same as in the previous figures ($\gamma_1 = 1.35$ and $\gamma_2 = 1.40$).

Statement 11 *In the case of a diffusive shock we have physically admissible shocks in the following cases:*

1. $c_0 < 0.5$, *in the interval* $c_0 < c < \bar{c}$ *(i.e. c jumps up),*
2. $c_0 > 0.5$, *in the interval* $\bar{c} < c < c_0$ *(i.e. c jumps down),*

where $\bar{c}$ *depends on* c_0. *In both cases we have the following jumps of the field variables:*

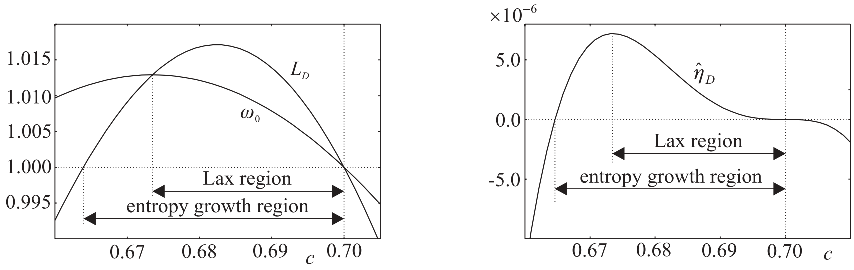

Fig. 4.7. Lax condition and entropy growth in a diffusive shock for $c_0 = 0.7$

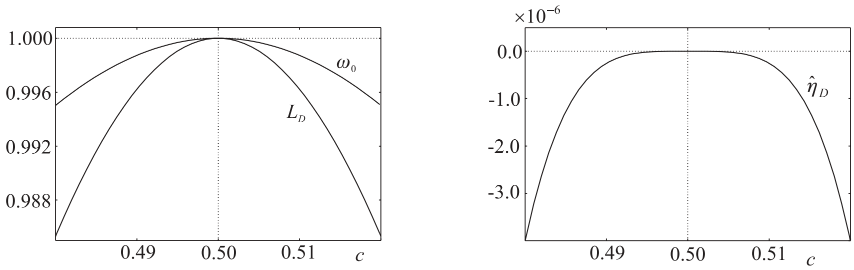

Fig. 4.8. Lax condition and entropy growth in a diffusive shock for $c_0 = 0.5$

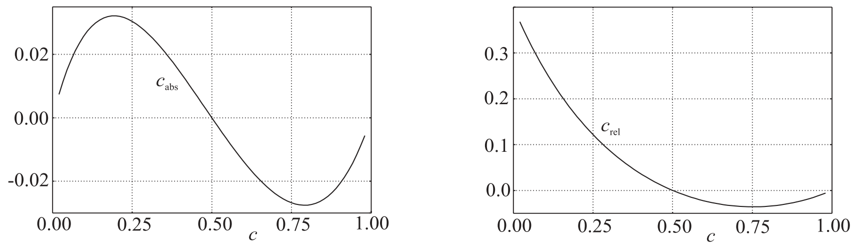

Fig. 4.9. Region of admissibility of a diffusive shock

– *pressure jumps down,*
– *density jumps down,*
– *velocity jumps up,*
– *temperature jumps down, and*
– *the jump of the diffusion flux is proportional to* $c - c_0$.

For $c_0 = c_0^* = 0.5$ *there are no admissible shocks.*

We remark that the diffusive shock in binary mixture of Euler fluids obeys the same properties as the second sound in a rigid heat conductor, discussed recently by Ruggeri et al. [16]. Using their terminology, the case $c_0 < 0.5$ corresponds to the *hot shock*, while the case $c_0 > 0.5$ corresponds to the *cold shock*. The terms hot and cold are associated with the jump of temperature in the rigid heat conductor, here corresponding to the appropriate jump of the concentration. The locally exceptional case $c_0 = c_0^* = 0.5$ corresponds to the appearance of the critical temperature in a rigid heat conductor. This apparent similarity of the results is not surprising. Ruggeri [5] showed that the governing equations (2.1) for the binary mixture of Euler fluids could be reduced under some suitable approximations to the set of equations for a rigid heat conductor.

In Fig. 10 we show the absolute ($c_{abs} = \bar{c} - c_0$) and the relative ($c_{rel} = (\bar{c} - c_0)/c_0$) width of the admissibility region as a function of c_0, where $\bar{c}$ is the value determined by the Liu condition. In fact we recall that, for the Liu theorem [20,21], the admissible region (in which the shock becomes stable) is what we have called in the previous figures Lax region. We observe also in this example that the criterion of a positive entropy growth predicts a larger interval with respect to the Liu condition.

Our analysis predicts the stability only for very weak shocks since the admissibility region, determined by c_{abs}, is very small. Strong diffusive shocks, with a jump of the concentration larger than c_{abs}, cannot persist and will be smoothed out immediately. Thus, we can conclude that the approach based on the principles of Extended Thermodynamics is in agreement with the classical Fick's theory only for strong diffusive shocks.

Case 3: Characteristic shock. The final case is when $s = \lambda^{(3)} = \lambda_0^{(3)} = 0$, which corresponds to the characteristic contact shock in the single fluid case. From the set of Rankine–Hugoniot equations (4.21) it is easy to obtain the following result.

Statement 12 *In the case of a binary mixture, there exists a characteristic shock moving with velocity* $s = \lambda^{(3)} = \lambda_0^{(3)} = 0$ *and propagating in an equilibrium state at rest. This shock is described by the following solution of the Rankine–Hugoniot equations:*

$$v = v_0 = 0, \quad c = c_0, \quad p = p_0, \quad J = J_0 = 0, \quad \rho - \textit{arbitrary}. \tag{4.53}$$

We observe that the contact shock wave does not exist if the wave is travelling in a non-equilibrium state with $J_0 \neq 0$ (see (4.21)).

5 Conclusion

In this paper we studied a binary mixture of Euler fluids from the point of view introduced in [5]. In this model the mixture is viewed as a single fluid with new fields describing the diffusion flux and the concentration of one constituent. In particular, we discussed one-dimensional nonlinear wave propagation into a region where the fluid is in a state of rest (or uniform rectilinear motion) and without diffusion flux. The results we obtained are in full accordance with the point of view put forward by the model.

We assumed from the outset that the difference between the atomic masses of the constituents is negligible. Through this assumption it is possible to make a clear distinction between the phenomena characteristic of the mixture as a whole and the phenomena that are peculiar to the constituents. Namely, we showed that in an unperturbed state the speeds of propagation and the corresponding eigenvectors can be split into two groups: one corresponds to the single fluid case and carries weak discontinuities in density, velocity, and temperature of the whole mixture, while the other corresponds to the so-called second sound and describes weak discontinuities of the concentration variable and the diffusion flux. We found that the critical times for the sound acceleration waves propagating with the greatest speed have the same form as in the single fluid case, provided that we introduce an average ratio of the specific heats that depends on the concentration. We also showed that the critical time for the mixture is bounded by the critical times for the constituents.

In the analysis of k-shocks, we showed that there exist three different types of shocks, two of them having formal counterparts in the single fluid case. The sonic shock, corresponding to the sound speed, carries the jumps of density, velocity, pressure, and temperature, while the concentration and the diffusion flux remain unaltered. This type of shock is admissible for supersonic values of the speed of propagation, just as in the case of a single fluid. There also exists a characteristic contact shock carrying only disturbances in density. Finally, we proved the existence of a diffusive shock, peculiar to the mixture, which corresponds to the speed of propagation of the second sound. For this shock we discovered the case of local exceptionality for the critical value of the concentration variable $c_0^* = 0.5$. In such a case, there are no admissible shocks. The physically admissible diffusive shocks appear for $c_0 < c < \bar{c}$ when $c_0 < 0.5$, and for $\bar{c} < c < c_0$ when $c_0 > 0.5$. These results are completely equivalent to the results obtained recently for the propagation of thermal disturbances in a rigid heat conductor.

In conclusion, we want to comment on two possible generalizations for future study. As was already discussed, these results are the first step towards the more realistic case of a binary mixture consisting of constituents with unequal masses. They are also an indicator of the appearance of a completely new feature – the local exceptionality – that could also be expected in the general case. On the other hand, we analyzed nonlinear waves in a binary mixture propagating in their own right, and assumed that the unperturbed state in front of the wave front is always an equilibrium state at rest or in uniform rectilinear motion. In reality, i.e. in the case of generic initial data for acceleration waves or for a general Riemann problem, the nonlinear waves propagate, in general, into a perturbed region. Therefore, further study could be directed towards a numerical study of the propagation of nonlinear waves in order to get deeper insight into this problem. Once again, the results obtained in this paper could serve as a starting point and they give a good hint about what one could expect in more complicated cases.

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