NON LINEAR WAVE PROPAGATION IN BINARY MIXTURES OF EULER FLUIDS*

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In this work we analyze propagation of non linear waves in mixtures of ideal Euler fluids. If the difference between molecular masses is negligible, we can separate the properties resembling the single fluid case from the ones peculiar to mixtures. We also showed that diffusive k-shock is locally exceptional.

1. Introduction

It is well-known that extended thermodynamic theory could be well applied to the study of the mixtures of Euler fluids, i.e. fluids which are neither viscous, nor heat-conducting (see Müller and Ruggeri[1]). Recently Ruggeri[2] observed that a binary mixture of Euler fluids could be seen as a single one heat-conducting fluid. Purpose of the present account is to give a brief survey of non linear wave propagation in such a medium. Namely, we shall see what are the main consequences of specific structure of the mathematical model on characteristic speeds and non linear waves, i.e. acceleration and shock waves.

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2. Mathematical Model and Basic Assumptions

Governing equations for the binary mixture of Euler fluid could be written in the form of a single fluid if the field variables describing behaviour of the mixture as a whole (density ρ, velocity $\mathbf{v}$ and temperature T) are extended with variables describing behaviour of one constituent (concentration variable c and diffusion flux vector $\mathbf{J}$). Here we shall analyze one-dimensional case without chemical reactions, thus reducing the balance laws to the form:

$$
\begin{aligned}
&\partial_t \rho + \partial_x(\rho v) = 0;\\
&\partial_t(\rho c) + \partial_x(\rho c v + J) = 0;\\
&\partial_t(\rho v) + \partial_x\left(\rho v^2 + p + \frac{J^2}{\rho c(1-c)}\right) = 0;\\
&\partial_t(\rho c v + J) + \partial_x\left(\rho c v^2 + 2vJ + \frac{J^2}{\rho c} + \nu\right) = -\beta(T)J;\\
&\partial_t\left(\frac{1}{2}\rho v^2 + \rho\varepsilon\right) + \partial_x\left(\left(\frac{1}{2}\rho v^2 + \rho\varepsilon + p\right)v + \frac{J^2}{\rho c(1-c)}v + \frac{1}{\alpha}J\right) = 0.
\end{aligned}
\tag{1}
$$

Governing equations consist of balance laws of mass, momentum and energy of the mixture (Eqs. $(1)_1$, $(1)_3$ and $(1)_5$) and balance laws of mass and momentum of one constituent (Eqs. $(1)_2$ and $(1)_4$). In (1) we used $p = p_1 + p_2$ for the total pressure and $\rho\varepsilon = \rho_1\varepsilon_1 + \rho_2\varepsilon_2$ for the total internal energy of the mixture, $\nu = p_1$ for the partial pressure of one constituent while:

$$
\frac{1}{\alpha} = \left(\varepsilon_1 + \frac{p_1}{\rho_1} + \frac{1}{2}u_1^2\right) - \left(\varepsilon_2 + \frac{p_2}{\rho_2} + \frac{1}{2}u_2^2\right). \tag{2}
$$

is the difference of enthalpies and $m_1 = -\beta(T)J$ represents the momentum exchange between constituents. We shall assume that constituents obey thermal and caloric equation of state of ideal gas $p_a = (k_B/m_a)\rho_a T$, $\varepsilon_a = p_a/(\rho_a(\gamma_a - 1))$, $a = 1, 2$, $k_B = 1.38 \cdot 10^{-23} J/K$ - Boltzmann constant, m_a - molecuar masses of constituents. These constitutive assumptions, which are in accordance with entropy principle, give to the system (1) the structure of a quasi-linear hyperbolic system of balance laws.

Following the same idea which led to specific structure of governing equations, we can derive unifying constitutive equations for the mixture $p = \hat{p}(\rho, c, T)$, $\varepsilon = \hat{\varepsilon}(c, T)$, provided an average atomic mass $m = \hat{m}(c)$ and average ratio of specific heats $\gamma = \hat{\gamma}(c)$ are introduced. General form of these functions could be found in the recent paper of Ruggeri and Simić[3].

Here we shall restrict our attention to the special case characterized by the *equal masses assumption*: $m = m_1 \approx m_2 = \text{const}$. It was shown[1] that this assumption leads to decoupling of the equations for linear wave propagation in binary mixtures. At the same time unifying constitutive equations are reduced to the following form:

$$p = k\rho T\,; \qquad \varepsilon_I = \frac{kT}{\gamma(c) - 1}\,; \tag{3}$$
$$\nu = p_1 = cp\,; \qquad \frac{1}{\gamma(c) - 1} = \frac{c}{\gamma_1 - 1} + \frac{1 - c}{\gamma_2 - 1}\,; \qquad k = \frac{k_B}{m}\,,$$

where ε_I is intrinsic internal energy and we shall assume $\gamma_1 < \gamma_2$.

3. Characteristic Speeds and Acceleration Waves

Although equal masses assumption could be criticized as too restrictive, and seems to be too close to the single fluid model, it gives a possibility for the qualitative analysis of the problem. Namely, if the system (1) is transformed into normal form $\partial_t \mathbf{u} + \hat{\mathbf{A}}(\mathbf{u})\partial_x \mathbf{u} = \hat{\mathbf{f}}(\mathbf{u})$ for field variables $\mathbf{u} = (\rho, c, v, J, T)^T$, characteristic equation $\det(\hat{\mathbf{A}} - \lambda \mathbf{I}) = 0$ is of the fifth degree with respect to characteristic speed:

$$\begin{aligned} &k\,T\,\rho^2\,(\gamma(c) - 1)\,\big\{(1 - 2c)\,J^3 - (1 - 5\,c\,(1 - c))\,\rho\,U\,J^2 \\ &\quad -2\,c\,(1 - c)\,(1 - 2\,c)\,\rho^2\,U^2\,J\, - c^2\,(1 - c)^2\,\rho^3\,U\,(U^2 - k\,T)\big\} \\ &\quad + \big(J^2 - 2\,(1 - c)\,\rho\,U\,J\, + (1 - c)^2\,\rho^2 (U^2 - k\,T)\big) \\ &\quad \times \big(J^2 - 2\,c\,\rho\,U\,J\, + c^2\,\rho^2 (U^2 - k\,T)\big)\,(\rho\,U + \Gamma\,J\,(\gamma(c) - 1)) = 0, \end{aligned} \tag{4}$$

where $U = v - \lambda$ and $\Gamma = 1/(\gamma_1 - 1) - 1/(\gamma_2 - 1)$.

In the sequel we shall analyze the wave propagation into a region where the mixture is at rest and without diffusion ($v_0 = 0,\ J_0 = 0$): $\mathbf{u}_0 = (\rho_0, c_0, 0, 0, T_0)^T$. This assumption simplifies the structure of Eq. (4) and leads to a conclusion that characteristic speeds in unperturbed state $\mathbf{u}_0$ could be splitted in two groups. First group has the same form as in the case of single Euler fluid:

$$\lambda_0^{(1)} = -\sqrt{\gamma_0\,k\,T_0}, \qquad \lambda_0^{(3)} = 0, \qquad \lambda_0^{(5)} = \sqrt{\gamma_0\,k\,T_0}, \tag{5}$$

but carries the information about the mixture through the average ratio of specific heats $\gamma_0 = \gamma(c_0)$. Second group, peculiar to mixture, corresponds to the propagation of the so-called *second sound*:

$$\lambda_0^{(2)} = -\sqrt{k\,T_0}, \qquad \lambda_0^{(4)} = \sqrt{k\,T_0}, \tag{6}$$

here related to changes in concentration and diffusion flux.

Propagation of the highest speed acceleration waves, which propagate along characteristic $\phi(x,t) = x - C_0 t = 0$, $C_0 = \sqrt{\gamma_0 k T_0}$, is based upon amplitude equation of Bernoulli type. As shown by Boillat[4] (see also Ruggeri[5]) it governs the behaviour of weak discontinuities for all hyperbolic systems of balance laws. It can be showed that critical times for the formation of shocks in horizontal and vertical direction, respectively:

$$t_{crt}(c_0) = \frac{2C_0}{G_0(\gamma_0+1)}\,; \quad t_{crt}(c_0) = \frac{2C_0}{g\gamma_0}\ln\left\{1 + \frac{g\gamma_0}{G_0(\gamma_0+1)}\right\}\,, \tag{7}$$

have the same form as in the single fluid case[5] when expressed in terms of initial acceleration jump $G_0 = [v_t](0) = [v_\phi]\phi_t(0)$. Moreover, they are bounded by the corresponding values of critical times of the constituents:

$$t^{(2)}_{crt} = t_{crt}(0) \leq t_{crt}(c_0) \leq t_{crt}(1) = t^{(1)}_{crt}. \tag{8}$$

4. Shock Waves

Analysis of shock waves in binary mixture of Euler fluids will be based upon the solution of Rankine-Hugoniot equations which govern the jump of field variables across the wave front:

$$\begin{aligned}
&[\rho u] = 0\,;\\
&[\rho c u - J] = 0\,;\\
&\left[\rho u^2 + p + \frac{J^2}{\rho c(1-c)}\right] = 0\,;\\
&\left[\rho c u^2 - 2Ju + \frac{J^2}{\rho c} + \nu\right] = 0\,;\\
&\left[\left(\frac{1}{2}\rho u^2 + \rho\varepsilon\right)u + pu + \frac{J^2}{\rho c(1-c)}u - \frac{1}{\alpha}J\right] = 0\,,
\end{aligned} \tag{9}$$

where $u = s - v$, s - speed of shock. Our attention will be restricted to k-shocks - weak shock waves which bifurcate from trivial solution of (9) where the speed of shock corresponds to the characteristic speed.

Along with the search for nontrivial solutions of Rankine-Hugoniot equations, a question of shock admissibility will be raised. Apart from the classical case of genuine nonlinearity ($\nabla_{\mathbf{u}}\lambda \cdot \mathbf{d} \neq 0$ for all $\mathbf{u}$), where Lax condition $\lambda_0 < s < \lambda$ and entropy growth criterion $\eta = -s[h^0] + [h] > 0$ could be well applied, and linearly degenerate case ($\nabla_{\mathbf{u}}\lambda \cdot \mathbf{d} = 0$ for all $\mathbf{u}$), where

$s = \lambda_0 = \lambda$ and $\eta \equiv 0$, we shall also encounter the case of *local exceptionality*. In this case condition of genuine nonlinearity is violated on some manifold in state space (so-called critical manifold). Consequently, Lax condition has to be substituted by more general Liu condition $s(\mathbf{u}_0, \bar{\mu}) \leq s(\mathbf{u}_0, \mu)$ for all $\mu_0 \leq \bar{\mu} \leq \mu$, and entropy growth criterion has to adjoined with the principle of superposition of shocks (see Liu and Ruggeri[6]). Therefore, in the sequel we shall employ the following expression for the entropy growth function across the shock $\eta = [\rho u S] - [\Psi]$:

$$S = \frac{k}{\gamma(c) - 1} \ln\left(\frac{p}{\rho^{\gamma(c)}}\right) - k\left\{c \ln c + (1-c)\ln(1-c)\right\}\,; \tag{10}$$

$$\Psi = kJ\left\{\Gamma \ln\left(\frac{p}{\rho}\right) - \ln\left(\frac{c}{1-c}\right)\right\}\,. \tag{11}$$

We can distinguish following three cases.

4.1. *Sonic Shock*

This case is characterized by the absence of jumps of concentration variable $c \equiv c_0$ and diffusion flux $J \equiv J_0 = 0$ and the following nontrivial solution of Rankine-Hugoniot equations:

$$\begin{aligned}
\pi &= \frac{p}{p_0} = \left(1 + \mu_0^2\right) M_0^2 - \mu_0^2\,; \\
w &= \frac{u}{u_0} = \frac{1}{r} = \frac{\rho_0}{\rho} = \frac{1}{M_0^2}\left\{1 - \mu_0^2\left(1 - M_0^2\right)\right\}\,; \\
\theta &= \frac{T}{T_0} = \frac{\pi}{r} = \pi w = \frac{1}{M_0^2}\left\{\left(1 + \mu_0^2\right) M_0^2 - \mu_0^2\right\}\left\{1 - \mu_0^2\left(1 - M_0^2\right)\right\}\,,
\end{aligned} \tag{12}$$

where $M_0 = u_0/C_0$ is sonic Mach number and $\mu^2(c) = \frac{\gamma(c)-1}{\gamma(c)+1}$, $\mu_0 = \mu(c_0)$. It bifurcates from the characeristic speed $\lambda_0^{(5)}$ - speed of sound, and obeys the very same properties of shock admissibility as in the single fluid case: it is genuinely nonlinear and admissible for $M_0 > 1$.

4.2. *Diffusive Shock*

In this case nontrivial solution of the system (9) can be expressed in terms of concentration c as shock parameter. It is goverened by the solution of biquadratic equation:

$$a_0(c)\omega_0^4 + a_1(c)\omega_0^2 + a_2(c) = 0, \tag{13}$$

where $\omega_0 = u_0/C_{0D}$, $C_{0D} = \sqrt{k\,T_0}$, is diffusive Mach number, and reads:

$$\begin{aligned}
&\omega_0 = \left\{ -\frac{a_1(c)}{2a_0(c)} - \sqrt{\left(\frac{a_1(c)}{2a_0(c)}\right)^2 - \frac{a_2(c)}{a_0(c)}} \right\}^{1/2}; \\
&w = \frac{1}{r} = \frac{1}{\alpha(c)}\,\frac{1+\omega_0^2(c)}{\omega_0^2(c)}; \\
&\pi = \left(1 - \frac{\beta(c)}{\alpha(c)}\right)\,(1+\omega_0^2(c)); \\
&\theta = \frac{\pi}{r} = \pi\, w = \frac{1}{\alpha(c)}\,\left(1 - \frac{\beta(c)}{\alpha(c)}\right)\,\frac{\left(1+\omega_0^2(c)\right)^2}{\omega_0^2(c)}; \\
&J = \rho_0\, C_{0D}\,\omega_0(c)\,(c - c_0),
\end{aligned} \tag{14}$$

where the explicit form of coefficients $a_0(c)$, $a_1(c)$, $a_2(c)$, $\alpha(c)$ and $\beta(c)$ will be omitted. It can be shown that solution (12) bifurcates from the second sound eigenspeed $\lambda_0^{(4)}$.

This case is particularly interesting since it obeys the property of local exceptionality. Namely, by a straightforward computation one can prove:

$$\nabla_{\mathbf{u}}\lambda^{(4)} \cdot \mathbf{d}^{(4)}\Big|_{\mathbf{u}_0} = \frac{1 - 2\,c_0}{c_0\,(1-c_0)}\,\sqrt{k\,T_0} \tag{15}$$

so that critical manifold has structure $\mathbf{u}_0 = (\rho_0, c_0, v_0, 0, T_0)$ for $c_0 = 0.5$ and arbitrary values of other field variables. This result strongly influences admissibility of diffusive shock in such a way that:

(i) if $c_0 < 0.5$ shocks are admissible for $c_0 < c < \bar{c}$;
(ii) if $c_0 > 0.5$ shocks are admissible for $\bar{c} < c < c_0$;
(iii) if $c_0 = 0.5$ there are no admissible shocks.

Numerical computation of Liu condition and entropy growth across the shock confirms this assertion (see Fig. 1). In fact, we compared dimensionless speed of diffusive shock L_D and diffusive Mach number ω_0. These results are in accordance with discussion of second sound phenomena in rigid heat conductor given by Ruggeri, Muracchini and Seccia[7].

4.3. *Characteristic Shock*

The final case is obtained for $s = \lambda^{(3)} = \lambda_0^{(3)} = 0$, and corresponds to the characteristic shock in the single fluid case. From the set of the Rankine-Hugoniot equations (9) it is easy to obtain the following result nontrivial solution:

$$v = v_0 = 0; \quad c = c_0; \quad p = p_0; \quad J = J_0 = 0; \quad \rho - \text{arbitrary}. \qquad (16)$$

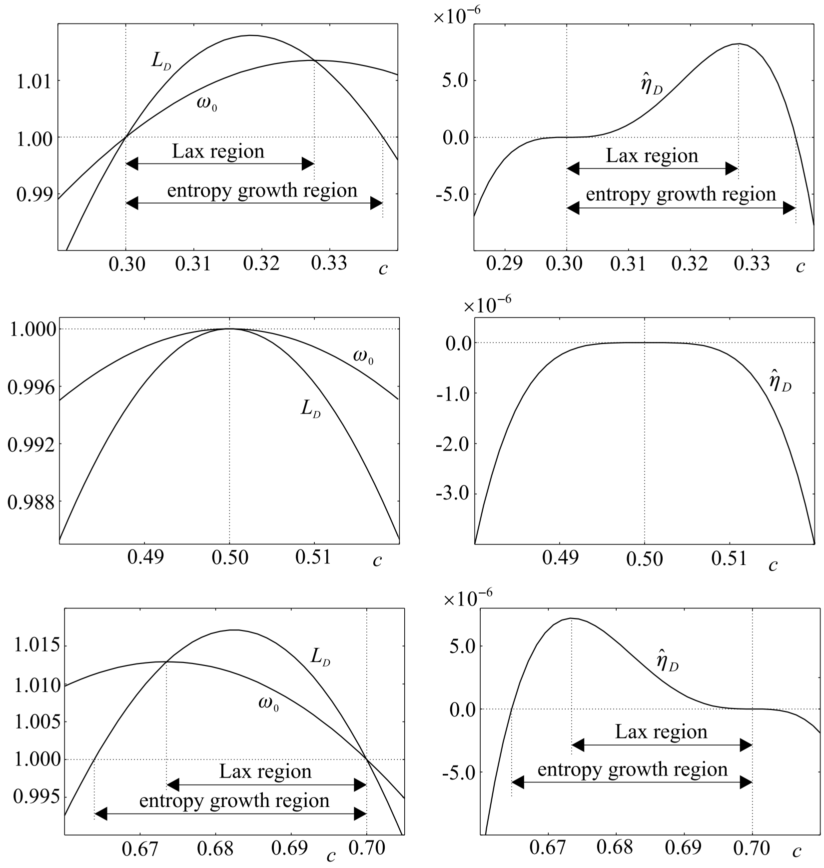

Figure 1. Lax condition and entropy growth in diffusive shock for $c_0 = 0.3$, $c_0 = 0.5$, $c_0 = 0.7$ and $\gamma_1 = 1.35$, $\gamma_2 = 1.40$

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