PERGAMON

International Journal of Non-Linear Mechanics 37 (2002) 197–211

INTERNATIONAL JOURNAL OF
NON-LINEAR
MECHANICS

www.elsevier.com/locate/ijnonlinmec

On the symmetry approach to polynomial conservation laws of one-dimensional Lagrangian systems

Srboljub S. Simic

Faculty of Engineering, Department of Applied Mechanics, University of Novi Sad, Trg Dositeja Obradovica 6, 21000 Novi Sad, Yugoslavia

Received 9 March 2000; received in revised form 25 July 2000; accepted 12 October 2000

Abstract

In this paper we analyze polynomial conservation laws of one-dimensional non-autonomous Lagrangian dynamical systems $\ddot{x} = -\partial\Pi(t,x)/\partial x$. The analysis is based upon application of Noether's theorem which relates the existence of conservation laws to the symmetries of Hamilton's action integral. It is shown that the existence of first integrals depends on the solution of the system of first-order partial differential equations — generalized Killing's equations. General solution of the problem is formally determined. It is demonstrated that the final form of dynamical system and corresponding conservation law depends on the solution of the so-called potential equation. However, the structure of symmetry transformations, which generate particular class of conservation laws, could be prescribed independent of the solution of potential equation. This fact is used to underline phenomenological aspect of symmetry approach. Its pragmatic value is confirmed through several concrete examples. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Conservation laws; Noether's theorem; Lagrangian systems

1. Introduction

Knowledge of conservation laws (first integrals, invariants, constants of motion) is of great importance in the study of dynamical systems. There is a deep belief that every conservation law reflects some profound physical principle acting in the system. On the other hand, sufficient number of independent first integrals leads to a complete integrability of dynamical system. Between these extremes (phenomenological and pragmatic) there is a variety of problems in which conservation laws play a very important role.

One of the most curious aspects of conservation laws is their relation to invariance properties of dynamical systems. Jacobi was already acquainted with the fact that conservation of linear and angular momentum come out by virtue of translational and rotational invariance in Euclidean sense (see Ref. [1]), while conservation of energy is an implication of the invariance with respect to time translation [2]. In a certain sense there exist a subtle relation between symmetry aspects of conservation laws and invariant properties of space and time in classical mechanics, i.e. their homogeneity and isotropy. These fundamental symmetries [1]

E-mail address: ssimic@uns.ns.ac.yu (S.S. Simic).

[1] Terms invariance and symmetry will be treated as synonyms throughout the paper.

PII: S0020-7462(00)00106-2

motivated the study of conservation laws from geometrical and group-theoretical point of view. Theorem of Emmy Noether [3] emanated as one of the most brilliant results of the symmetry approach to the problem. Most of the results concerning Noetherian approach to conservation laws of classical mechanics could be found in the papers of Hill [4] and Desloge and Karch [5]. However, it is not the phenomenological aspect which is the only one covered by Noether's theorem. It turned out that it could be used as a reliable tool for derivation of new conservation laws of dynamical systems. Lots of examples of application of Noether's theorem could be found in Ref. [6].

Although apparent physical meaning of classical conservation laws deeply influenced investigations in this field, intriguing problems of classical mechanics, engineering and contemporary physics motivated the study of the new types of constants of motion. As paradigmatic ones we could mention first integral of Sophia Kowalevskaya for the motion of gyroscope (see Ref. [7]), and infinite set of polynomial constants of motion for the Toda-lattice [8]. It have to be admitted that most of the analyses of polynomial conservation laws were based upon mathematical considerations, rather than physical ones. It is also hard to give any a posteriori interpretation for such conservation laws.

In this paper we intend to perform a study of polynomial conservation laws of one-dimensional dynamical systems. To be precise, we shall restrict ourselves to the study of Lagrangian dynamical systems whose behavior could be described by a single function of independent variable-time t, and state variables — generalized coordinate x and generalized velocity $\dot{x}$, Lagrangian function L. Furthermore, we shall assume that it possesses the structure of kinetic potential

$$L(t,x,\dot{x}) = \tfrac{1}{2}\dot{x}^2 - \Pi(t,x), \tag{1.1}$$

where $\Pi(t,x)$ denotes the potential of the system and an overdot denotes differentiation with respect to an independent variable. In that case, dynamical equation which have the structure of Euler–Lagrange differential equation reduces to the following form:

$$\ddot{x} = -\partial\Pi(t,x)/\partial x. \tag{1.2}$$

As it is well-known, lots of processes in mechanics and physics have mathematical models which could be reduced to a single equation of the form (1.2). Therefore, primary aim of our analysis is to establish a general procedure for the study of polynomial conservation laws of dynamical system (1.2) and to determine what restrictions have to be made on the potential of the system in order that it admits the existence of the first integrals of such a type. Since the analysis will be based on Noetherian approach, a study of the structure of infinitesimal transformations will also be performed. Finally, several illustrative examples will be given. Differentiability of functions is assumed to be of sufficiently high order, and summation with respect to repeated indices is assumed throughout the paper.

2. The method

A variety of methods have been developed for the search of conservation laws, for example method of integrating factors, also termed as direct or ad hoc procedure (Refs. [9–11]), method of similarity variables [12] and transformation approach of Crespo da Silva [13]. Most of these studies are devoted either to dynamical systems of particular structure, or to derivation of the first integrals of a special type. On the other hand, within the group-theoretical approach certain procedures of considerable generality have been established. They relate the existence of first integrals to the symmetries of certain mathematical object which serves for the description of dynamical system. Several recent results concerned with the symmetry aspects of Lagrangian and Hamiltonian formalism are contained in Refs. [14–16]. In the following text we shall give a brief overview of the Noether's theorem, which also falls into this category.

Let us suppose that the state of the dynamical system at any instant of time is determined by n-tuple of generalized coordinates $\mathbf{x}(t) = (x^1(t), \ldots, x^n(t))$ and n-tuple of generalized velocities $\dot{\mathbf{x}}(t) = (\dot{x}^1(t), \ldots, \dot{x}^n(t))$, and that behavior of the system could be described by a Lagrangian function $L(t,\mathbf{x},\dot{\mathbf{x}})$. Under these assumptions differential equations of motion have the structure of

Euler–Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = 0, \quad i = 1,\ldots,n \tag{2.1}$$

which come out as a consequence of the necessary condition for extremum of the Hamilton's action integral

$$J = \int_{t_0}^{t_1} L(t, \mathbf{x}(t), \dot{\mathbf{x}}(t))\ \mathrm{d}t. \tag{2.2}$$

Let us introduce in our analysis infinitesimal transformations of time t and generalized coordinates $\mathbf{x}$ to another set of independent and dependent variables $(t, \mathbf{x}) \to (T(\varepsilon), \mathbf{X}(\varepsilon))$, where ε is a small real parameter, such that $(T(0), \mathbf{X}(0)) = (t, \mathbf{x})$. Because of their local character these transformations could be completely described by a generator of transformations and its prolongation. Since we are tending to construct general polynomial conservation laws for dynamical systems described by a quadratic Lagrangian (1.1), it will be appropriate to use so-called Lie-Bäcklund tangent transformations [17] or generalized symmetries [18]. Contrary to the classical Lie point symmetries, whose generators could depend only on time t and generalized coordinates $\mathbf{x}$, generators of generalized symmetries could be the functions of generalized velocities or even higher order derivatives. In the present study we shall confine ourselves with the following structure of generator $\mathbf{v}$ and its first prolongation $\mathrm{pr}^{(1)}\mathbf{v}$

$$\mathbf{v} = \tau(t, \mathbf{x}, \dot{\mathbf{x}})\frac{\partial}{\partial t} + \xi^i(t, \mathbf{x}, \dot{\mathbf{x}})\frac{\partial}{\partial x^i},$$

$$\mathrm{pr}^{(1)}\mathbf{v} = \mathbf{v} + \eta^i(t, \mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}})\frac{\partial}{\partial \dot{x}^i},$$

$$\eta^i(t, \mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}) = \dot{\xi}^i - \dot{x}^i\dot{\tau}, \quad i = 1,\ldots,n. \tag{2.3}$$

In the context of Lagrangian mechanics these transformations were first introduced by Djukic [19]. Condition of invariance of Hamilton's action integral (2.2) with respect to transformations (2.3) could be reduced to the following functional relation:

$$L\dot{\tau} + \frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial x^i}\xi^i + \frac{\partial L}{\partial \dot{x}^i}(\dot{\xi}^i - \dot{x}^i\dot{\tau}) = \dot{f}, \tag{2.4}$$

where $f(t, \mathbf{x}, \dot{\mathbf{x}})$ is a gauge function. If such a condition is satisfied, it is said that transformations (2.3) represent symmetries of the action integral. Eq. (2.4) is known as basic identity [6] and Noether–Bessel–Hagen equation [20], and action integral is said to be absolute invariant if $f(t, \mathbf{x}, \dot{\mathbf{x}}) = 0$. Noether's theorem states that to every symmetry of the action integral (2.2) there corresponds a conservation law of Lagrangian dynamical system (2.1) in the form

$$L\tau + \frac{\partial L}{\partial \dot{x}^i}(\xi^i - \dot{x}^i\tau) - f = \text{const}. \tag{2.5}$$

Our choice of the method is motivated by the fact that Noether's theorem tackles both aspects which are of considerable importance in the study of conservation laws. On phenomenological level it sheds light on direct connection of conservation laws of classical mechanics to the symmetries of space and time (see Refs. [5,6]). On the other hand, it also possesses a pragmatic value since it could be used as a reliable tool for derivation of new first integrals. Using classical point symmetries Vujanovic [21] established a procedure for obtaining group generators and gauge function. This procedure is based on decomposition of the basic identity (2.4) into a set of partial differential equations of the first order — generalized Killing's equations. This idea could be easily exploited in our analysis under the assumption of the regularity of Lagrangian function ($\det(\partial^2 L/\partial \dot{x}^i \partial \dot{x}^j) \neq 0$) which enables us to resolve Eqs. (2.1) with respect to generalized accelerations, and to calculate basic identity along the trajectory of dynamical system. Other possible interpretations of Noether's theorem could be found in the paper of Sarlet and Cantrijn [22].

3. General results

In this section we shall offer some results of general nature. As far as author is aware, there have been only few attempts to generally investigate polynomial conservation laws of one-dimensional systems. Lewis and Leach [23] studied polynomial invariants of Hamiltonian systems with time-dependent potential from the point of view of direct method, and Sarlet [24] performed a similar study using method of integrating factors. However, they were not concerned with detailed analysis of higher-degree invariants. Particular results concerned with higher-degree polynomial invariants of Emden–Fowler equation are contained in the papers of Airault [25,26] We intend to give an overview of polynomial conservation laws using a procedure outlined in previous paragraph. At the same time, we shall keep in our mind restrictions on the structure of dynamical system and Lagrangian function described in the Introduction. Namely, we shall seek the conservation laws of one-dimensional dynamical system (1.2), whose Lagrangian is given in Eq. (1.1), which have polynomial form with respect to generalized velocity, i.e.

$$I = \sum_{k=0}^{n} a_k(t,x)\dot{x}^k = \text{const.}, \tag{3.1}$$

where functions $a_k(t,x)$, $k = 0,1,\dots,n$, have to be determined in the course of analysis. Desired form of the first integral, in conjunction with the assumed structure of Lagrangian function (1.1) and general form of conservation law in Noetherian approach (2.5), suggests that generators of generalized symmetry transformations have to be taken in polynomial form

$$\xi(t,x,\dot{x}) = \sum_{k=0}^{n-1} \xi_k(t,x)\dot{x}^k, \quad \tau(t,x,\dot{x}) = \sum_{k=0}^{n-2} \tau_k(t,x)\dot{x}^k,$$
$$f(t,x,\dot{x}) = \sum_{k=0}^{n-1} f_k(t,x)\dot{x}^k. \tag{3.2}$$

By substituting Eq. (3.2) into (2.5) we obtain the general form of conservation law

$$I = \sum_{k=0}^{n} [\xi_{k-1}(t,x) - \tfrac{1}{2}\tau_{k-2}(t,x) - \Pi(t,x)\tau_k(t,x) - f_k(t,x)]\dot{x}^k = \text{const.} \tag{3.3}$$

where following restrictions are assumed

$$\xi_i(t,x) = 0 \text{ and } f_i(t,x) = 0 \quad \text{for } i < 0 \text{ and } i > n-1;$$
$$\tau_j(t,x) = 0 \quad \text{for } j < 0 \text{ and } j > n-2. \tag{3.4}$$

Basic identity (2.4), calculated along trajectory of dynamical system (1.2), is then reduced to

$$\sum_{k=0}^{n+1} \Psi_k(t,x)\,\dot{x}^k = 0 \tag{3.5}$$

where coefficients $\Psi_k(t,x)$, $k = 0, 1, \dots, n+1$, could be expressed in terms of generators of symmetry transformations

$$\begin{aligned}\Psi_k(t,x) = {} & \frac{\partial}{\partial x}[\xi_{k-2}(t,x) - \tfrac{1}{2}\tau_{k-3}(t,x) \\ & - \Pi(t,x)\tau_{k-1}(t,x) - f_{k-1}(t,x)] \\ & + \frac{\partial}{\partial t}[\xi_{k-1}(t,x) - \tfrac{1}{2}\tau_{k-2}(t,x) \\ & - \Pi(t,x)\tau_k(t,x) - f_k(t,x)] \\ & - (k+1)[\xi_k(t,x) - \tfrac{1}{2}\tau_{k-1}(t,x) \\ & - \Pi(t,x)\tau_{k+1}(t,x) - f_{k+1}(t,x)]\frac{\partial \Pi(t,x)}{\partial x}.\end{aligned} \tag{3.6}$$

Since coefficients $\Psi_k(t,x)$ are velocity independent, condition of invariance (3.5) of the action integral could be satisfied identically if the following system of equations holds

$$\Psi_k(t,x) = 0, \quad k = 0, 1, \dots, n+1. \tag{3.7}$$

System (3.7) consists of first-order partial differential equations. In accordance with adopted terminology, they represent the system of generalized Killing's equations. In order to clarify forthcoming analysis a few remarks concerning the process of solving this problem are ought to be given. First, we have to notice that there are no constraints, such as initial or boundary conditions, imposed to the system (3.7). This implies that we are free to use its general, complete or even particular solution for determination of symmetry transformations and corresponding conservation law. Second, we have to be very cautious in discussing the solution of the system of generalized Killing's equations since it possesses trivial solution ($\xi = \tau = f = 0$) which leads to a trivial conservation law $I = 0$. There are also some pathological situations in which non-trivial solution of system (3.7) leads to a trivial first integral, i.e. mathematical constant. These cases, most of which occur when $\xi^i - \dot{x}^i\tau = 0$, have to be avoided, if possible. Finally, it is easy to recognize that generalized Killing's equations form the system of $n+2$ equations with $3n$ unknown functions $\xi_i(t,x)$, $f_i(t,x)$ $(i = 0,1,\ldots,n-1)$, $\tau_j(t,x)$ $(j = 0,1,\cdots,n-2)$ and $\Pi(t,x)$. Obviously, this system is not determined and we could deliberately include certain assumptions concerning the structure of these functions in order to create fully determined system. This procedure would have not be performed in unified manner. Since its correct application sometimes needs skill and experience, it is useful to notice the relation between the coefficients in the expression for the first integral (3.1) and generators of symmetry transformations which reads

$$a_k(t,x) = \xi_{k-1}(t,x) - \tfrac{1}{2}\tau_{k-2}(t,x) - \Pi(t,x)\tau_k(t,x) - f_k(t,x); \quad k = 0,1,\ldots,n, \tag{3.8}$$

where restrictions (3.4) are assumed. Now, if we treat $a_i(t,x)$, $i = 0,1,\ldots,n$, as unknown functions in system (3.7), then it represents the system of $n+2$ equations with $n+2$ unknown functions $a_i(t,x)$ and $\Pi(t,x)$. This analysis facilitates the process of integration of generalized Killing's equations (3.7). One of their features is the possibility of solving by recursion. Namely, we can start with the equation for the highest index $k = n+1$, and find its general solution

$$\Psi_{n+1}(t,x) = \frac{\partial a_n(t,x)}{\partial x} = 0 \Rightarrow a_n(t,x) = \xi_{n-1}(t,x) - \tfrac{1}{2}\tau_{n-2}(t,x) = \theta(t) \tag{3.9}$$

where $\theta(t)$ is an arbitrary function of time. Using this result we can find general solution of Eq. (3.7) for $k = n$

$$\Psi_n(t,x) = \frac{\partial a_{n-1}(t,x)}{\partial x} + \frac{\partial a_n(t,x)}{\partial t} = 0 \Rightarrow a_{n-1}(t,x) = \xi_{n-2}(t,x) - \tfrac{1}{2}\tau_{n-3}(t,x) - f_{n-1}(t,x) = -\dot{\theta}(t)\,x + \varphi(t) \tag{3.10}$$

where $\varphi(t)$ is an arbitrary function of time. Further recursive integration could be performed in the same manner, but only formally since in Eqs. (3.7) for $k < n$ figures potential $\Pi(t,x)$ which is not determined. However, a rather obvious conclusion could be drawn out from this analysis. Generalized Killing's equations with indices $1 \leqslant k \leqslant n+1$ determine the coefficients in expression (3.1) for the conservation law, while the equation $\Psi_0(t,x) = 0$ elicits functional relation which serves for determination of the potential, which will throughout the paper be referred to as *potential equation*. Although this is not the only way of coping the problem of integration of the system (3.7), it is nevertheless conceptually transparent, applicable without any regard to the degree of conservation law and it leads to a potential equation of the lowest possible order.

4. Solutions of generalized Killing's equations

Let us turn our attention to certain particular systems of generalized Killing's equations, i.e. polynomial conservation laws of prescribed degree. We shall give a detailed study of the solution to system (3.7) for conservation laws of degree $n \leqslant 4$.

4.1. Linear conservation laws

According to the structure of generators of symmetry transformations (3.2) and restrictions (3.4), in the case $n = 1$ conservation law (3.3) has the form

$$I = \xi_0(t,x)\dot{x} - f_0(t,x) = \text{const.} \tag{4.1}$$

and generalized Killing's equations (3.7) reduce to

$$\frac{\partial \xi_0}{\partial x} = 0, \tag{4.2a}$$

$$\frac{\partial f_0}{\partial x} - \frac{\partial \xi_0}{\partial t} = 0, \tag{4.2b}$$

$$\frac{\partial f_0}{\partial t} + \xi_0\frac{\partial \Pi}{\partial x} = 0. \tag{4.2c}$$

It is easy to determine general solution of Eqs. (4.2a) and (4.2b)

$$\xi_0(t,x) = \theta(t), \tag{4.3a}$$

$$f_0(t,x) = \dot{\theta}(t)x + \varphi(t), \tag{4.3b}$$

where $\theta(t)$ and $\varphi(t)$ are arbitrary functions of time. By substituting (4.3) in (4.2c) we obtain potential equation for linear conservation laws

$$\dddot{\theta}(t)x + \ddot{\varphi}(t) + \theta(t)\frac{\partial \Pi}{\partial x} = 0. \tag{4.4}$$

General solution of Eq. (4.4) reveals the general form of the potential $\Pi(t,x)$ which admits the existence of linear conservation law. Rather simple form of potential equation gives us an opportunity to actually obtain its general solution which reads

$$\Pi(t,x) = -\frac{1}{2}\frac{\dddot{\theta}(t)}{\theta(t)}x^2 - \frac{\ddot{\varphi}(t)}{\theta(t)}x + \alpha(t) \tag{4.5}$$

where $\alpha(t)$ is an arbitrary function of time. Thus, we can conclude that dynamical system of the form

$$\ddot{x} = \frac{\dddot{\theta}(t)}{\theta(t)}x + \frac{\ddot{\varphi}(t)}{\theta(t)} \tag{4.6}$$

possesses linear conservation law

$$I = \theta(t)\dot{x} - \dot{\theta}(t)x - \varphi(t) = \text{const.} \tag{4.7}$$

An interesting result appeared in this analysis. In the class of non-autonomous one-dimensional Lagrangian systems only linear ones could have linear first integrals. Although linear conservation laws are usually connected with ignorable coordinates, it is easy to see that this is not the case here.

4.2. *Quadratic conservation laws*

In accordance with Noetherian approach general form of quadratic first integral is

$$\begin{aligned} I = {} & [\xi_1(t,x) - \tfrac{1}{2}\tau_0(t,x)]\dot{x}^2 \\ & + [\xi_0(t,x) - f_1(t,x)]\dot{x} \\ & - [\Pi(t,x)\tau_0(t,x) + f_0(t,x)] = \text{const.}, \end{aligned} \tag{4.8}$$

while generalized Killing's equations are reduced to

$$\frac{\partial}{\partial x}[\xi_1 - \tfrac{1}{2}\tau_0] = 0, \tag{4.9a}$$

$$\frac{\partial}{\partial x}[\xi_0 - f_1] + \frac{\partial}{\partial t}[\xi_1 - \tfrac{1}{2}\tau_0] = 0, \tag{4.9b}$$

$$\frac{\partial}{\partial x}[\Pi\tau_0 + f_0] - \frac{\partial}{\partial t}[\xi_0 - f_1] + 2[\xi_1 - \tfrac{1}{2}\tau_0]\frac{\partial \Pi}{\partial x} = 0, \tag{4.9c}$$

$$\frac{\partial}{\partial t}[\Pi\tau_0 + f_0] + [\xi_0 - f_1]\frac{\partial \Pi}{\partial x} = 0. \tag{4.9d}$$

Successive integration of Eqs. (4.9a)–(4.9c) gives the following general solution:

$$\xi_1(t,x) - \tfrac{1}{2}\tau_0(t,x) = \theta(t), \tag{4.10a}$$

$$\xi_0(t,x) - f_1(t,x) = -\dot{\theta}(t)x + \varphi(t), \tag{4.10b}$$

$$\Pi(t,x)\tau_0(t,x) + f_0(t,x) = -\tfrac{1}{2}\ddot{\theta}(t)x^2 + \dot{\varphi}(t)x - 2\theta(t)[\Pi(t,x) + \psi(t)], \tag{4.10c}$$

where $\theta(t)$, $\varphi(t)$ and $\psi(t)$ are arbitrary functions of time. Substitution of (4.10b) and (4.10c) into (4.9d) leads to a potential equation for quadratic invariants:

$$\tfrac{1}{2}\dddot{\theta}(t)x^2 - \ddot{\varphi}(t)x + 2\frac{\partial}{\partial t}[\theta(t)\Pi(t,x)] + 2\frac{\mathrm{d}}{\mathrm{d}t}[\theta(t)\psi(t)] + [\dot{\theta}(t)x - \varphi(t)]\frac{\partial \Pi(t,x)}{\partial x} = 0. \tag{4.11}$$

A brief inspection of potential equation (4.11) shows that it falls into a category of linear partial differential equations of the first order. As it is well-known, a standard procedure exists for their solution. In our case, arbitrariness of functions $\theta(t)$, $\varphi(t)$ and $\psi(t)$ implies great complexity of potential equation (4.11) and narrows the space for finding its general solution. However, under certain conditions general solution can be found. Some of these conditions were examined in Ref. [27] within the problem of generalization of Lewis' invariant.

Analysis of quadratic invariants clarifies the role of potential equation. Namely, it serves for determination of the potential $\Pi(t,x)$ and arbitrary functions of time, obtained in the course of integration of generalized Killing's equations, in such way that it is satisfied identically. Once these functions are obtained the structure of dynamical system and corresponding conservation law could be determined.

4.3. Cubic conservation laws

General form of cubic invariants derived from Eq. (3.3) is

$$I = [\xi_2(t,x) - \tfrac{1}{2}\tau_1(t,x)]\dot{x}^3 + [\xi_1(t,x) - \tfrac{1}{2}\tau_0(t,x) - f_2(t,x)]\dot{x}^2 + [\xi_0(t,x) - \Pi(t,x)\tau_1(t,x) - f_1(t,x)]\dot{x} - [\Pi(t,x)\tau_0(t,x) + f_0(t,x)] = \text{const.}, \quad (4.12)$$

For the sake of brevity, we shall not give the explicit form of generalized Killing's equations for cubic invariants, but rather write down their solution

$$\xi_2(t,x) - \tfrac{1}{2}\tau_1(t,x) = \theta(t), \quad (4.13a)$$

$$\xi_1(t,x) - \tfrac{1}{2}\tau_0(t,x) - f_2(t,x) = -\dot{\theta}(t)x + \varphi(t), \quad (4.13b)$$

$$\xi_0(t,x) - \Pi(t,x)\tau_1(t,x) - f_1(t,x) = \tfrac{1}{2}\ddot{\theta}(t)x^2 - \dot{\varphi}(t)x + 3\theta(t)[\Pi(t,x) + \psi(t)], \quad (4.13c)$$

$$\Pi(t,x)\tau_0(t,x) + f_0(t,x) = \tfrac{1}{6}\dddot{\theta}(t)x^3 - \tfrac{1}{2}\ddot{\varphi}(t)x^2 + 3\frac{d}{dt}[\theta(t)\psi(t)]x + 2\dot{\theta}(t)\int x\frac{\partial\Pi(t,x)}{\partial x}\,dx + 3\int\frac{\partial}{\partial t}[\theta(t)\Pi(t,x)]\,dx - 2\varphi(t)[\Pi(t,x) + \kappa(t)] \quad (4.13d)$$

where $\theta(t)$, $\varphi(t)$, $\psi(t)$ and $\kappa(t)$ are arbitrary functions of time. Corresponding potential equation reads

$$\frac{1}{6}\theta^{IV}(t)x^3 - \frac{1}{2}\dddot{\varphi}(t)x^2 + 3\frac{d^2}{dt^2}[\theta(t)\psi(t)]x - 2\frac{d}{dt}[\varphi(t)\kappa(t)] + 3\frac{\partial}{\partial t}\int\frac{\partial}{\partial t}[\theta(t)\Pi(t,x)]\,dx + 2\frac{\partial}{\partial t}\left[\dot{\theta}(t)\int x\frac{\partial\Pi(t,x)}{\partial x}\,dx\right] - 2\frac{\partial}{\partial t}[\varphi(t)\Pi(t,x)] + \left[\frac{1}{2}\ddot{\theta}(t)x^2 - \dot{\varphi}(t)x + 3\theta(t)\psi(t) + 3\theta(t)\Pi(t,x)\right]\frac{\partial\Pi(t,x)}{\partial x} = 0. \quad (4.14)$$

These equations were carefully studied in Ref. [28].

It is easily seen that the structure of the potential equation (4.14) is much more complex than in the cases of linear and quadratic first integrals since it possesses integro-differential form. Process of its solution is much relied on guess-work. One of the possibilities is to propose in advance the form of the potential $\Pi(t,x)$ and then try to find the solution of potential equation in that class of functions. This procedure was widely employed in Ref. [28].

4.4. Quartic conservation laws

In final part of this section we shall examine conservation laws which are of the fourth degree (quartic) with respect to generalized velocity. According to the results developed in previous section, general form of quartic invariant is

$$I = [\xi_3(t,x) - \tfrac{1}{2}\tau_2(t,x)]\dot{x}^4 + [\xi_2(t,x) - \tfrac{1}{2}\tau_1(t,x) - f_3(t,x)]\dot{x}^3 + [\xi_1(t,x) - \tfrac{1}{2}\tau_0(t,x) - \Pi(t,x)\tau_2(t,x) - f_2(t,x)]\dot{x}^2 + [\xi_0(t,x) - \Pi(t,x)\tau_1(t,x) - f_1(t,x)]\dot{x} - [\Pi(t,x)\tau_0(t,x) + f_0(t,x)] = \text{const.} \quad (4.15)$$

At the same time, solution of the generalized Killing's equations reads

$$\xi_3(t,x) - \tfrac{1}{2}\tau_2(t,x) = \theta(t), \quad (4.16a)$$

$$\xi_2(t,x) - \tfrac{1}{2}\tau_1(t,x) - f_3(t,x) = -\dot{\theta}(t)x + \varphi(t), \quad (4.16b)$$

$$\xi_1(t,x) - \tfrac{1}{2}\tau_0(t,x) - \Pi(t,x)\tau_2(t,x) - f_2(t,x) = \tfrac{1}{2}\ddot{\theta}(t)x^2 - \dot{\varphi}(t)x + 4\theta(t)[\Pi(t,x) + \psi(t)], \quad (4.16c)$$

$$\xi_0(t,x) - \Pi(t,x)\tau_1(t,x) - f_1(t,x)$$

$$= -\frac{1}{6}\dddot{\theta}(t)x^3 + \frac{1}{2}\ddot{\varphi}(t)x^2 - 4\frac{d}{dt}[\theta(t)\psi(t)]x$$

$$- 4\int\frac{\partial}{\partial t}[\theta(t)\Pi(t,x)]\,dx$$

$$- 3\dot{\theta}(t)\int x\frac{\partial\Pi(t,x)}{\partial x}\,dx$$

$$+ 3\varphi(t)[\Pi(t,x) + \kappa(t)], \qquad (4.16d)$$

$$\Pi(t,x)\tau_0(t,x) + f_0(t,x) =$$

$$-\frac{1}{24}\theta^{IV}(t)x^4 + \frac{1}{6}\bar{\varphi}(t)x^3 - 2\frac{d^2}{dt^2}[\theta(t)\psi(t)]x^2$$

$$+ 3\frac{d}{dt}[\varphi(t)\kappa(t)]x - 4\int\left[\frac{\partial}{\partial t}\int\frac{\partial}{\partial t}[\theta(t)\Pi(t,x)]\,dx\right]dx$$

$$- 3\int\frac{\partial}{\partial t}\left[\dot{\theta}(t)\int x\frac{\partial\Pi(t,x)}{\partial x}\,dx\right]dx$$

$$+ 3\int\frac{\partial}{\partial t}[\varphi(t)\Pi(t,x)]\,dx$$

$$- \ddot{\theta}(t)\int x^2\frac{\partial\Pi(t,x)}{\partial x}\,dx$$

$$+ 2\dot{\varphi}(t)\int x\frac{\partial\Pi(t,x)}{\partial x}\,dx$$

$$- 8\theta(t)\int\Pi(t,x)\frac{\partial\Pi(t,x)}{\partial x}\,dx$$

$$- 8\theta(t)\psi(t)[\Pi(t,x) + \nu(t)], \qquad (4.16e)$$

while the potential equation have the following form:

$$\frac{1}{24}\theta^{V}(t)x^4 - \frac{1}{6}\varphi^{IV}(t)x^3 + 2\frac{d^3}{dt^3}[\theta(t)\psi(t)]x^2$$

$$- 3\frac{d^2}{dt^2}[\varphi(t)\kappa(t)]x + 8\frac{d}{dt}[\theta(t)\psi(t)\nu(t)]$$

$$+ 4\frac{\partial}{\partial t}\int\left[\frac{\partial}{\partial t}\int\frac{\partial}{\partial t}[\theta(t)\Pi(t,x)]\,dx\right]dx$$

$$+ 3\frac{\partial}{\partial t}\int\frac{\partial}{\partial t}\left[\dot{\theta}(t)\int x\frac{\partial\Pi(t,x)}{\partial x}\,dx\right]dx$$

$$- 3\frac{\partial}{\partial t}\int\frac{\partial}{\partial t}[\varphi(t)\Pi(t,x)]\,dx$$

$$+ \frac{\partial}{\partial t}\left[\ddot{\theta}(t)\int x^2\frac{\partial\Pi(t,x)}{\partial x}\,dx\right]$$

$$- 2\frac{\partial}{\partial t}\left[\dot{\varphi}(t)\int x\frac{\partial\Pi(t,x)}{\partial x}\,dx\right]$$

$$+ 8\frac{\partial}{\partial t}\left[\theta(t)\int\Pi(t,x)\frac{\partial\Pi(t,x)}{\partial x}\,dx\right]$$

$$+ 8\frac{\partial}{\partial t}[\theta(t)\psi(t)\Pi(t,x)]$$

$$+ \left\{\frac{1}{6}\dddot{\theta}(t)x^3 - \frac{1}{2}\ddot{\varphi}(t)x^2 + 4\frac{d}{dt}[\theta(t)\psi(t)]x\right.$$

$$+ 4\int\frac{\partial}{\partial t}[\theta(t)\Pi(t,x)]\,dx$$

$$+ 3\dot{\theta}(t)\int x\frac{\partial\Pi(t,x)}{\partial x}\,dx - 3\varphi(t)[\Pi(t,x)$$

$$\left. + \kappa(t)]\right\}\frac{\partial\Pi(t,x)}{\partial x} = 0. \qquad (4.17)$$

5. Structure of the symmetry transformations

Noether's theorem revealed, in a quite impressive manner, that paradigmatic linear and quadratic conservation laws of classical mechanics are tightly connected to the symmetries of space and time. Namely, conservation of linear and angular momentum are consequences of translational and rotational invariances in Euclidean sense, while conservation of energy is related to invariance with respect to time translation. In this section we intend to show that pragmatic side of Noether's theorem could be connected to phenomenological aspects of the symmetry transformations. We shall investigate the structure of the symmetry transformations which yield polynomial first integrals of system (1.2), and to analyze their relation to the classical symmetries of paradigmatic conservation laws. Moreover, certain features of the symmetries which generate higher-degree invariants will be discussed.

This study is motivated by the fact that solutions of the generalized Killing's equations reveal only the structure of the coefficients $a_i(t,x)$ in first integral (3.1) whose final form depends on the solution of the potential equation. On the other hand the structure of the symmetry is not exactly determined, i.e. different symmetries could lead to the same first integral. This freedom of choice enables us to propose in advance the structure of the symmetry transformations which will imply the existence of the conservation law of desired form, without any regard to the final solution of the potential equation.

5.1. Invariance with respect to generalized space and time translations

In the analysis of linear conservation laws we have obtained general solution of generalized Killing's equations (Eq. (4.3)) which yield following group generator and gauge function:

$$\mathbf{v} = \theta(t)\frac{\partial}{\partial x}, \qquad f = \dot{\theta}(t)x + \varphi(t). \tag{5.1}$$

It is worth to note that Eq. (5.1) was derived without any additional assumptions. In this case group generator describes coordinate translation in a generalized sense since the generator of the coordinate transformation is an explicit function of time. Thus, it could be said that linear first integrals come out as a consequence of the gauge invariance of an action integral with respect to a generalized coordinate translation. This statement corresponds to the well-known property of the integral of linear momentum.

Analysis of quadratic conservation laws is a bit more complex and makes it possible to obtain different generators for the same first integral. Since we tend to establish correspondence with symmetry transformations of classical energy integral, we shall assume that there is no transformation of generalized coordinate, i.e.

$$\xi_0(t,x) = 0, \qquad \xi_1(t,x) = 0 \;\Rightarrow\; \xi(t,x,\dot{x}) = 0. \tag{5.2}$$

Under this assumption we can derive group generator and gauge function in a unique way from Eq. (4.10)

$$\mathbf{v} = -\,2\theta(t)\frac{\partial}{\partial t}, \quad f = [\dot{\theta}(t)x - \varphi(t)]\dot{x} - \tfrac{1}{2}\ddot{\theta}(t)x^2 + \dot{\varphi}(t)x - 2\theta(t)[\Pi(t,x) + \psi(t)]. \tag{5.3}$$

It is now inevitable that quadratic first integrals of dynamical system (1.2) are implied by time translation in a generalized sense described above.

We can see that results presented in Eqs. (5.1) and (5.3) are totally independent of the solution of corresponding potential equations (4.4) and (4.11), respectively. Thus, our conclusion concerning the relation of linear and quadratic conservation laws to generalized coordinate and time translation is valid for any conservation law of system (1.2) which falls into these categories.

5.2. Autogenerative character of conservation laws

Let us turn ourselves to the analysis of cubic and quartic invariants. In the case of cubic invariants, from solution (4.13) of generalized Killing's equations we can derive special class of symmetry transformations under the assumption that there is no transformation of independent variable, i.e.

$$\tau(t,x,\dot{x}) = 0 \;\Leftrightarrow\; \tau_0(t,x) = 0; \quad \tau_1(t,x) = 0. \tag{5.4}$$

Furthermore, we could assume that $\xi_1(t,x) = 0$ and that following relation holds:

$$\xi_0(t,x) = -\,2\theta(t)\Pi(t,x). \tag{5.5}$$

Under these assumptions generator of symmetry transformations reduces to

$$\mathbf{v} = 2\theta(t)L(t,x,\dot{x})\frac{\partial}{\partial x}, \tag{5.6}$$

while gauge function could be uniquely determined from Eq. (4.13). Since its structure is too complex it will not be given here explicitly. This result was already presented in Ref. [28], where twofold purpose of Lagrangian function was emphasized for the first time. On the one hand, it describes the behavior of the system through the Euler–Lagrange differential equation, while on the other serves for the construction of conservation laws of the same system since it makes a deep influence on the structure of symmetry transformations.

Generators of symmetry transformations of the fourth-degree first integrals could be studied in a similar way. Namely, analysis of solution (4.16) of generalized Killing's equations shows that we can assume that generalized coordinate does not suffer any transformation

$$\xi(t,x,\dot{x}) = 0 \Leftrightarrow \xi_i(t,x) = 0, \quad i = 0,1,2,3. \tag{5.7}$$

If, in addition, we suppose that following relations hold:

$$\tau_0(t,x) = 4\theta(t)\Pi(t,x), \quad \tau_1(t,x) = 0, \tag{5.8}$$

then it is easy to deduce from Eq. (4.16) the structure of infinitesimal generator

$$\mathbf{v} = -\,4\theta(t)L(t,x,\dot{x})\frac{\partial}{\partial t}. \tag{5.9}$$

As in the case of cubic invariants, explicit form of gauge function, which can be uniquely determined from Eq. (4.16), will be omitted for its complexity. Again, we can observe that Lagrangian function of the system could play a fundamental role in the study of the symmetries of the problem. Generator of time transformation consists of Lagrangian of the system multiplied by an arbitrary function of time. The possibility for this kind of analysis of the fourth degree first integrals was anticipated in the paper of Vujanovic et al. [29] in the study of the generalized Emden–Fowler equation.

Outstanding property of cubic and quartic invariants, obtained in conjunction with symmetry approach, was in Ref. [28] termed as *autogenerative character of conservation laws*. This phrase tends to emphasize that these first integrals are generated by the same Lagrangian function which serves for the description of the dynamics of the system, as it was mentioned before. Finally, it must be noted that symmetry transformations obtained here do not depend on the solution of the corresponding potential equations, as in the cases of linear and quadratic invariants.

6. Examples

Previous sections were devoted to the analysis of generalized Killing's equations and part of their solution which determines coefficients in the expression for first integral and symmetry transformations. In this section we shall be concerned with the potential equations whose solution determines the structure of the system and enables us to write down explicitly a corresponding conservation law.

Example 1. Let us consider linear first integrals of non-homogeneous Euler's differential equation, which could be derived from Eq. (4.6) for $\theta(t) = t^2$. Thus, using Eqs. (4.6) and (4.7) we can conclude that dynamical system

$$\ddot{x} = 2t^{-2}x + t^{-2}\dot{\varphi}(t) \tag{6.1}$$

have linear conservation law

$$I = t^2\dot{x} - 2tx - \varphi(t) = \text{const.}, \tag{6.2}$$

where $\varphi(t)$ is an arbitrary function of time. This result could be of considerable interest when we discuss the possibility of reduction of higher-degree first integrals to linear ones.

Example 2. Let us analyze potential equation (4.11) for quadratic conservation laws. At first, we shall assume that potential is time separable, i.e.

$$\Pi(t,x) = q(t)p(x). \tag{6.3}$$

Further, we shall propose the following structure for the functions obtained in the course of integration

$$\psi(t) = 0, \quad \theta(t) = 1,$$

$$\ddot{\varphi}(t) = 0 \;\Rightarrow\; \varphi(t) = 2at + b, \tag{6.4}$$

where a and b are arbitrary real constants. Under these assumptions Eq. (4.11) reduces to the following form:

$$2\dot{q}(t)p(x) - \varphi(t)q(t)\frac{\mathrm{d}p(x)}{\mathrm{d}x} = 0, \tag{6.5}$$

which is amenable to be solved by the method of separation of variables

$$\frac{2\dot{q}(t)}{\varphi(t)q(t)} = \frac{\mathrm{d}p(x)/\mathrm{d}x}{p(x)} = -k = \text{const.} \tag{6.6}$$

Simple integration gives the following result which reveals the structure of potential (6.3):

$$q(t) = \lambda\ \exp[-\tfrac{k}{2}(at^2 + bt + c)], \quad p(x) = \mathrm{e}^{-kx}, \tag{6.7}$$

where λ and c are arbitrary real constants. Thus, we can conclude from Eqs. (1.2), (4.8) and (6.3) that dynamical system

$$\ddot{x} = \lambda k\ \exp[-\tfrac{k}{2}(at^2 + bt + c)]\mathrm{e}^{-kx} \tag{6.8}$$

have quadratic conservation law of the form

$$I = \dot{x}^2 + (2at + b)\dot{x} - 2ax + 2\lambda\ \exp[-\tfrac{k}{2}(at^2 + bt + c)]\mathrm{e}^{-kx} = \text{const.} \tag{6.9}$$

It could be said that potential obtained in this analysis represents an one-dimensional time-dependent Toda-type potential.

Example 3. In this example we shall restrict ourselves to the cubic invariants of linear dynamical systems, i.e. systems with the potential

$$\Pi(t,x) = \tfrac{1}{2}q(t)x^2. \tag{6.10}$$

In order to simplify the analysis we shall assume that all functions obtained in the course of integration of generalized Killing's equations, except $\theta(t)$, are equal to zero

$$\varphi(t) = \psi(t) = \kappa(t) = 0. \tag{6.11}$$

According to these assumptions, potential equation (4.15) could be reduced to a single ordinary differential equation of the following form:

$$\theta^{IV}(t) + 10q(t)\ddot{\theta}(t) + 10\dot{q}(t)\dot{\theta}(t) + 3[\ddot{q}(t) + 3q^2(t)]\theta(t) = 0. \tag{6.12}$$

Since $q(t)$ is arbitrary function of time, it is very hard to obtain general solution of Eq. (6.12). Therefore, further assumptions have to be introduced to obtain solutions for certain particular classes of function $q(t)$. Here, we shall assume that functions $\theta(t)$ and $q(t)$ are power-law functions

$$\theta(t) = \theta_0 t^a; \quad q(t) = \lambda t^b, \tag{6.13}$$

where θ_0, λ, a and b are constants which are to be determined. By substituting Eq. (6.13) into Eq. (6.12) we obtain

$$(a^4 - 6a^3 + 11a^2 - 6a)t^{a-4} + 9\lambda^2 t^{a+2b} + (10\lambda a^2 - 10\lambda a + 10\lambda ab + 3\lambda b^2 - 3\lambda b)t^{a+b-2} = 0, \tag{6.14}$$

where θ_0 is omitted because of homogeneity. Since we wish to obtain solution of Eq. (6.14) for arbitrary value of t, it have to be transformed into homogeneous form with respect to time variable by imposing additional restrictions to constants a and b

$$a - 4 = a + 2b = a + b - 2. \tag{6.15}$$

From Eq. (6.15) we obtain $b = -2$, while constant a still remains undetermined. Under these conditions Eq. (6.14) is reduced to the following time-independent form:

$$a^4 - 6a^3 + (10\lambda + 11)a^2 - (30\lambda + 6)a + 9\lambda^2 + 18\lambda = 0. \tag{6.16}$$

If we suppose that value of parameter λ is proposed, then four distinct solutions could be obtained for constant a

$$a_1 = \tfrac{3}{2} + \tfrac{1}{2}\sqrt{1 - 4\lambda},$$
$$a_2 = \tfrac{3}{2} - \tfrac{1}{2}\sqrt{1 - 4\lambda},$$
$$a_3 = \tfrac{3}{2} + \tfrac{3}{2}\sqrt{1 - 4\lambda},$$
$$a_4 = \tfrac{3}{2} - \tfrac{3}{2}\sqrt{1 - 4\lambda}. \tag{6.17}$$

From the analysis performed in this example we can draw out a conclusion that dynamical system

$$\ddot{x} = -\lambda t^{-2}x \tag{6.18}$$

possesses following four cubic conservation laws, which correspond to the values of constant a given in Eq. (6.17)

$$I_1 = t^{(3/2)+(1/2)\sqrt{1-4\lambda}}\dot{x}^3 - (\tfrac{3}{2} + \tfrac{1}{2}\sqrt{1 - 4\lambda})t^{(1/2)+(1/2)\sqrt{1-4\lambda}}x\dot{x}^2$$

$$+\tfrac{1}{2}(1+2\lambda+\sqrt{1-4\lambda})t^{-(1/2)+(1/2)\sqrt{1-4\lambda}}x^2\dot{x}$$
$$-\tfrac{1}{2}\lambda(1+\sqrt{1-4\lambda})t^{-(3/2)+(1/2)\sqrt{1-4\lambda}}x^3$$
$$=\text{const.}; \tag{6.19a}$$

$$I_2 = t^{(3/2)-(1/2)\sqrt{1-4\lambda}}\dot{x}^3$$
$$-(\tfrac{3}{2}-\tfrac{1}{2}\sqrt{1-4\lambda})t^{(1/2)-(1/2)\sqrt{1-4\lambda}}x\dot{x}^2$$
$$+\tfrac{1}{2}(1+2\lambda-\sqrt{1-4\lambda})t^{-(1/2)-(1/2)\sqrt{1-4\lambda}}x^2\dot{x}$$
$$-\tfrac{1}{2}\lambda(1-\sqrt{1-4\lambda})t^{-(3/2)-(1/2)\sqrt{1-4\lambda}}x^3$$
$$=\text{const.}; \tag{6.19b}$$

$$I_3 = t^{(3/2)+(3/2)\sqrt{1-4\lambda}}\dot{x}^3$$
$$-(\tfrac{3}{2}+\tfrac{3}{2}\sqrt{1-4\lambda})t^{(1/2)+(3/2)\sqrt{1-4\lambda}}x\dot{x}^2$$
$$+\tfrac{3}{2}(1-2\lambda+\sqrt{1-4\lambda})t^{-(1/2)+(3/2)\sqrt{1-4\lambda}}x^2\dot{x}$$
$$-\tfrac{1}{2}(1-3\lambda+(1-\lambda)\sqrt{1-4\lambda})t^{-(3/2)+(3/2)\sqrt{1-4\lambda}}x^3$$
$$=\text{const.}; \tag{6.19c}$$

$$I_4 = t^{(3/2)-(3/2)\sqrt{1-4\lambda}}\dot{x}^3$$
$$-(\tfrac{3}{2}-\tfrac{3}{2}\sqrt{1-4\lambda})t^{(1/2)-(3/2)\sqrt{1-4\lambda}}x\dot{x}^2$$
$$+\tfrac{3}{2}(1-2\lambda-\sqrt{1-4\lambda})t^{-(1/2)-(3/2)\sqrt{1-4\lambda}}x^2\dot{x}$$
$$-\tfrac{1}{2}(1-3\lambda-(1-\lambda)\sqrt{1-4\lambda})t^{-(3/2)-(3/2)\sqrt{1-4\lambda}}x^3$$
$$=\text{const.} \tag{6.19d}$$

Since we are seeking for real cubic constants of motion, it have to be noted that they exist if and only if $\lambda < 1/4$. For these values of λ Euler's differential equation (6.18) have non-periodic solutions. It is interesting to notice that for $\lambda = -2$ first integral (6.19c) could be reduced to a linear one

$$I_3 = (t^2\dot{x} - 2tx)^3 = \text{const.}, \tag{6.20}$$

which comes out from our first example for $\varphi(t) = 0$. As far as the author is concerned, question of reducibility of higher-degree conservation laws to lower-degree ones does not have definite answer, i.e. there has not been established a general procedure for identification of reducible conservation laws, and each result have to be studied separately.

Example 4. In this example we shall try to obtain fourth-degree conservation laws for non-linear dynamical system of the following form:

$$\ddot{x} + q(t)x + r(t)x^n = 0, \tag{6.21}$$

whose potential is

$$\Pi(t,x) = \tfrac{1}{2}q(t)x^2 + r(t)\frac{x^{n+1}}{n+1}, \quad n \neq -1. \tag{6.22}$$

Dynamical system (6.21) represents a non-autonomous form of the well-known Duffing's equation. In order to simplify potential equation (4.17) we shall introduce the following assumptions concerning functions obtained in the course of integration:

$$\varphi(t) = \psi(t) = \kappa(t) = \nu(t) = 0. \tag{6.23}$$

Under these assumptions potential equation reduces to the polynomial with respect to x, with time-dependent coefficients. It will be satisfied identically if the coefficients were equal to zero simultaneously, which leads to decomposition of potential equation into three ordinary differential equations

$$\frac{1}{24}\theta^{V}(t) + \frac{1}{6}\frac{\mathrm{d}^3}{\mathrm{d}t^3}[\theta(t)q(t)] + \frac{1}{4}\frac{\mathrm{d}^2}{\mathrm{d}t^2}[\dot{\theta}(t)q(t)]$$
$$+\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}[\ddot{\theta}(t)q(t)] + \frac{\mathrm{d}}{\mathrm{d}t}[\theta(t)q^2(t)]$$
$$+\frac{1}{6}\dddot{\theta}(t)q(t) + \frac{2}{3}q(t)\frac{\mathrm{d}}{\mathrm{d}t}[\theta(t)q(t)]$$
$$+\dot{\theta}(t)q^2(t) = 0, \tag{6.24a}$$

$$\frac{4}{(n+1)(n+2)(n+3)}\frac{\mathrm{d}^3}{\mathrm{d}t^3}[\theta(t)r(t)]$$
$$+\frac{3}{(n+2)(n+3)}\frac{\mathrm{d}^2}{\mathrm{d}t^2}[\dot{\theta}(t)r(t)]$$
$$+\frac{1}{n+3}\frac{\mathrm{d}}{\mathrm{d}t}[\ddot{\theta}(t)r(t)]$$
$$+\frac{1}{6}\dddot{\theta}(t)r(t) + \frac{4}{n+1}\frac{\mathrm{d}}{\mathrm{d}t}[\theta(t)q(t)r(t)]$$
$$+\left\{\frac{2}{3}\frac{\mathrm{d}}{\mathrm{d}t}[\theta(t)q(t)] + \dot{\theta}(t)q(t)\right\}r(t)$$

$$+\frac{1}{n+2}\left\{\frac{4}{n+1}\frac{\mathrm{d}}{\mathrm{d}t}[\theta(t)r(t)]\right.$$
$$\left.+3\dot{\theta}(t)r(t)\right\}q(t)=0, \tag{6.24b}$$

$$\frac{4}{(n+1)^2}\frac{\mathrm{d}}{\mathrm{d}t}[\theta(t)r^2(t)]$$
$$+\frac{4}{(n+1)(n+2)}r(t)\frac{\mathrm{d}}{\mathrm{d}t}[\theta(t)r(t)]$$
$$+\frac{3}{n+2}\dot{\theta}(t)r^2(t)=0. \tag{6.24c}$$

At first, we can see that Eq. (6.24c), which could be integrated, yields the following relation:

$$r(t)=\lambda\theta^{-(n+3)/4}(t), \tag{6.25}$$

where λ is a constant of integration. In fact, Eq. (6.25) enables us to reduce system (6.24) to two ordinary differential equations with two unknown functions $\theta(t)$ and $q(t)$. Since we tend to obtain any solution of these equations, we shall restrict the analysis to power-law ones, as in the previous example

$$\theta(t)=\theta_0 t^a, \quad q(t)=\mu t^b, \tag{6.26}$$

where θ_0, μ, a and b are constants to be determined. Without loss of generality we can assume $\theta_0=1$. Like in the case of cubic invariants, Eq. (6.24a) will determine the structure of linear part of dynamical system (6.21). From the condition of homogeneity with respect to time variable we obtain $b=-2$. As in the previous example, time-independent form of Eq. (6.24a) enables us to determine admissible values of parameter a, if we assume that μ is proposed

$$a_1=2,$$
$$a_2=2+\sqrt{1-4\mu},$$
$$a_3=2-\sqrt{1-4\mu},$$
$$a_4=2+2\sqrt{1-4\mu},$$
$$a_5=2-2\sqrt{1-4\mu}. \tag{6.27}$$

With this result Eq. (6.24b) reduces to a time-independent form which reads

$$(a^3-6a^2+8a+16a\mu-32\mu)n+7a^3-42a^2$$
$$+56a+112a\mu-224\mu=0. \tag{6.28}$$

For different values of parameter a chosen from Eq. (6.27) we obtain corresponding values of parameter n which admits the existence of quartic first integral. If $a=a_1=2$, Eq. (6.28) will be satisfied identically for arbitrary value of n. If $a=a_i (i=2,3,4,5)$, from Eq. (6.28) we obtain $n=-7$. Thus, we can conclude that in the first case dynamical system

$$\ddot{x}+\mu t^{-2}x+\lambda t^{-(n+3)/2}x^n=0 \tag{6.29}$$

have following first integral:

$$I=t^2\dot{x}^4-2tx\dot{x}^3$$
$$+\left(x^2+2\mu x^2+4\lambda t^{-(n-1)/2}\frac{x^{n+1}}{n+1}\right)\dot{x}^2$$
$$-\left(2\mu t^{-1}x^3+4\lambda t^{-(n+1)/2}\frac{x^{n+2}}{n+1}\right)\dot{x}$$
$$+\mu^2t^{-2}x^4+4\lambda\mu t^{-(n+3)/2}\frac{x^{n+1}}{n+1}$$
$$+4\lambda^2t^{-(n+1)}\frac{x^{2(n+1)}}{(n+1)^2}=\text{const.} \tag{6.30}$$

It is noteworthy that this first integral could be reduced to a quadratic one since $I=K^2$, where

$$K=t\dot{x}^2-x\dot{x}+\mu t^{-1}x^2$$
$$+2\lambda t^{-(n+1)/2}\frac{x^{n+1}}{n+1}=\text{const.} \tag{6.31}$$

In four other cases $a=a_i (i=2,3,4,5)$ dynamical system

$$\ddot{x}+\mu t^{-2}x+\lambda t^a x^{-7}=0 \tag{6.32}$$

have fourth-degree conservation law of the following form:

$$I=t^a\dot{x}^4-at^{a-1}x\dot{x}^3$$
$$+\{[2\mu+\tfrac{1}{2}a(a-1)]t^{a-2}x^2$$
$$-\tfrac{2}{3}\lambda t^{2a}x^{-6}\}\dot{x}^2$$

$$
\begin{aligned}
&-\tfrac{1}{3}\{[\mu(5a-4)\\
&+\tfrac{1}{2}a(a-1)(a-2)]t^{a-3}x^3\\
&-\lambda a t^{2a-1}x^{-5}\}\dot{x}\\
&+\tfrac{1}{24}[24\mu-44a\mu+16a^2\mu+24\mu^2\\
&+a(a-1)(a-2)(a-3)]t^{a-4}x^4\\
&-\tfrac{1}{12}\lambda[8\mu+a(a-2)]t^{2a-2}x^{-4}\\
&+\tfrac{1}{9}\lambda^2 t^{3a}x^{-12}=\text{const.} \qquad (6.33)
\end{aligned}
$$

This first integral cannot be reduced to a lower-degree one in general case, i.e. for arbitrary value of parameter μ.

7. Conclusions

In this paper we have studied polynomial conservation laws of one-dimensional Lagrangian systems whose Lagrangian function have the structure of kinetic potential. The analysis was founded on Noether's theorem which relates the existence of conservation laws to invariance properties of Hamilton's action integral. By virtue of this method we came into a position to consider both important aspects of the problem — phenomenological and pragmatic. These are the main results.

In general case, generators of symmetry transformations have to have polynomial structure with respect to generalized velocity, and thus fall into a category of Lie–Bäcklund tangent transformations or generalized symmetries. It was shown that the existence of polynomial first integrals depends on the solution of the system of generalized Killing's equations which form a set of first-order partial differential equations. Recursion procedure enabled us to formally resolve the problem, i.e. determine the most general form of the coefficients in the expression for the first integral and derive the potential equation whose solution explicitly determines the structure of dynamical system and corresponding conservation law. In this paper, first integrals of degree $n \leqslant 4$ were thoroughly discussed. It was discovered that the structure of symmetry transformations, which yield conservation laws of particular degree, could be proposed in advance, without any regard to the final solution of the potential equation. In the cases of linear and quadratic first integrals they correspond to coordinate and time translation in a generalized sense. Cubic and quartic first integrals came out as a consequence of symmetry transformations whose structure is determined by Lagrangian function of the system. Tempting to emphasize twofold purpose of Lagrangian function in these cases we stated that cubic and quartic conservation laws possess autogenerative character. Finally, in order to demonstrate pragmatic value of our investigations we have presented four illustrative examples which show different possibilities for the solution of potential equations. As a concluding remark we can express our opinion, which became apparent throughout the paper, that symmetry approach to the analysis of polynomial invariants is indeed a powerful one. It gives us an opportunity to resolve concrete problems, while on the other side retains its phenomenological flavor and enables us to make a breakthrough in the analysis of symmetry transformations of non-autonomous systems.

Acknowledgements

This paper represents a part of the author's Ph.D. thesis prepared under the guidance of Professor Djordje S. Djukic. Author expresses his gratitude to Professor Djukic and Professor Bozidar D. Vujanovic for help and encouragement during the development of the study.

References

[1] R.M.F. Houtappel, H. van Dam, E.P. Wigner, The conceptual basis and use of the geometrical invariance principles, Rev. Modern Phys. 37 (4) (1965) 595–632.
[2] L.D. Landau, E.M. Lifschitz, Mechanics, Nauka, Moscow, 1988 (in Russian).
[3] E. Noether, Invariante variationsprobleme, Nachr. Ges. Wiss. (1918) 235–257.
[4] E.L. Hill, Hamilton's Principle and the conservation theorems of mathematical physics, Rev. Modern Phys. 23 (3) (1951) 253–260.
[5] E.A. Desloge, R.I. Karch, Noether's theorem in classical mechanics, Am. J. Phys. 45 (4) (1977) 336–339.

[6] B.D. Vujanovic, S.E. Jones, Variational Methods in Nonconservative Phenomena, Academic Press, Boston, 1989.

[7] E.T. Whittaker, A Treatise on the Analytical Dynamics of Particles and Rigid Bodies, Cambridge University Press, London, 1965.

[8] M. Toda, Theory of Nonlinear Lattices, Springer, Berlin, 1981.

[9] W. Sarlet, L.Y. Bahar, A direct construction of first integrals for certain non-linear dynamical systems, Int. J. Non-Linear Mech. 15 (1980) 133–146.

[10] P.V. Ranganathan, Integrals of the Emden-Fowler equations, Int. J. Non-Linear Mech. 27 (4) (1992) 583–590.

[11] Dj.S. Djukic, T. Sutela, Integrating factors and conservation laws for nonconservative dynamical systems, Int. J. Non-Linear Mech. 19 (4) (1984) 331–339.

[12] S.E. Jones, W.F. Ames, Similarity variables and first integrals for ordinary differential equations, Int. J. Non-Linear Mech. 2 (1967) 257–260.

[13] M.R.M. Crespo da Silva, A transformation approach for finding first integrals of motion of dynamical systems, Int. J. Non-Linear Mech. 9 (1974) 241–250.

[14] G.H. Katzin, J. Levine, A gauge invariant formulation of time-dependent dynamical symmetry mappings and associated constants of motion for Lagrangian particle mechanics. I, J. Math. Phys. 17 (7) (1976) 1345–1350.

[15] A.S. Fokas, Group theoretical aspects of constants of motion and separable solutions in classical mechanics, J. Math. Anal. Appl. 68 (1979) 347–370.

[16] A.S. Fokas, P.A. Lagerstrom, Quadratic and cubic invariants in classical mechanics, J. Math. Anal. Appl. 74 (1980) 325–341.

[17] N.H. Ibragimov, Lie-Bäcklund Tangent Transformations, J. Math. Anal. Appl. 59 (1977) 145–162.

[18] P.J. Olver, Applications of Lie Groups to Differential Equations, 2nd Edition, Springer, New York, 1993.

[19] Dj.S. Djukic, A procedure for finding first integrals of mechanical systems with gauge-variant Lagrangians, Int. J. Non-Linear Mech. 8 (1973) 479–488.

[20] Dj.S. Djukic, B.D. Vujanovic, Noether's theory in classical nonconservative mechanics, Acta Mech. 23 (1975) 17–27.

[21] B. Vujanovic, A group-variational procedure for finding first integrals of dynamical systems, Int. J. Non-Linear Mech. 5 (1970) 269–278.

[22] W. Sarlet, F. Cantrijn, Generalizations of Noether's theorem in classical mechanics, SIAM Rev. 23 (4) (1981) 467–494.

[23] H.R. Lewis, P.G.L. Leach, A direct approach to finding exact invariants for one-dimensional time-dependent classical Hamiltonians, J. Math. Phys. 23 (12) (1982) 2371–2374.

[24] W. Sarlet, First integrals for one-dimensional particle motion in a nonlinear, time-dependent potential field, Int. J. Non-Linear Mech. 18 (4) (1983) 259–268.

[25] H. Airault, Existence and construction of quadratic invariants for the equation $(d^2/d\xi^2)v = \lambda(\xi)(d/d\xi)v + F(\xi, v)$, Int. J. Non-Linear Mech. 21 (3) (1986) 197–203.

[26] H. Airault, Polynomial invariants for the equation $(d^2/d\xi^2)v = \lambda(\xi)(d/d\xi)v + F(\xi, v)$, Int. J. Non-Linear Mech. 21 (4) (1986) 241–247.

[27] S.S. Simic, A note on generalization of the Lewis invariant and the Ermakov systems, J. Phys. A: Math. Gen., 33 (2000) 5435–5447.

[28] S.S. Simic, Cubic invariants of one-dimensional Lagrangian systems, Int. J. Non-Linear Mech. 35 (2) (2000) 333–345.

[29] B.D. Vujanovic, A.M. Strauss, S.E. Jones, P.P. Gillis, Polynomial conservation laws of the generalized Emden-Fowler equation, Int. J. Non-Linear Mech. 33 (2) (1998) 377–384.