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ON NOETHERIAN APPROACH TO INTEGRABLE CASES OF THE MOTION OF HEAVY TOP

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A b s t r a c t. *The paper is concerned with the problem of finding conservation laws for the motion of heavy top via Noether's theorem. Study of conservation laws is based upon application of Noether's theorem expressed in quasi-coordinates. It is shown that infinitesimal transformations have, in certain cases, clear geometric interpretation. The status of the conservation laws of particular character within this framework is also clarified.*

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1. *Introduction*

Along with the problem of three bodies, motion of the rigid body about a fixed point is one of the most intriguing and most inspirative problems of classical mechanics. A lot of new and fruitful ideas originated from this problem. On the other hand, the power of existing methods and approaches was often assessed through application to the analysis of the motion of the rigid body. Since 1750, when Euler derived basic equations and studied their solution, we witnessed an endless stream of literature concerned with this problem. The analyses diverged into several directions because the motion of the rigid body about a fixed point has both mathematical and practical significance. This paper tends to shed light on some aspects of the problem which arise in connection with the so-called symmetry approach.

By a heavy top we shall assume a rigid body which could move about a fixed point in a gravitational field. In order to describe the position of the body it is convenient to use two distinct frames of reference with the fixed point used as a common origin. One of them $Oxyz$ is fixed in space, while the other $O\xi\eta\zeta$ is fixed in the body. Location of the moving frame relative to a fixed one completely determines the position of the body. For the sake of simplicity, moving axes are chosen to coincide with the principal axes of inertia of the body. Let us denote by A, B and C moments of inertia of the rigid body for the axes $O\xi$, $O\eta$ and $O\zeta$, respectively, and by ξ_C, η_C and ζ_C coordinates of the center of mass with respect to a moving frame. Then, angular velocity $\boldsymbol{\omega}$, angular momentum $\mathbf{L}_O$, position vector $\mathbf{r}_C$ of the center of mass, and unit vector $\mathbf{k}$ of the vertical Oz axis could be expressed with respect to a moving frame as follows

$$\begin{aligned} \boldsymbol{\omega} &= \omega^1\mathbf{e}_1 + \omega^2\mathbf{e}_2 + \omega^3\mathbf{e}_3, \\ \mathbf{L}_O &= A\omega^1\mathbf{e}_1 + B\omega^2\mathbf{e}_2 + C\omega^3\mathbf{e}_3, \\ \mathbf{r}_C &= \xi_C\mathbf{e}_1 + \eta_C\mathbf{e}_2 + \zeta_C\mathbf{e}_3, \\ \mathbf{k} &= \alpha\mathbf{e}_1 + \beta\mathbf{e}_2 + \gamma\mathbf{e}_3, \end{aligned} \tag{1}$$

where $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are unit vectors of the axes $O\xi$, $O\eta$ and $O\zeta$, respectively. Basic dynamical equations for the motion of heavy top are derived from the angular momentum theorem $d\mathbf{L}_O/dt = \mathbf{M}_O$, where $\mathbf{M}_O$ denotes the moment of external forces with respect to a fixed point O. Since $\mathbf{M}_O = \mathbf{r}_C \times m\mathbf{g}$, where m is mass of the body, and $\mathbf{g} = -g\mathbf{k}$ is gravity acceleration, we obtain a set of Euler's dynamical equations

$$\begin{aligned} A\frac{d\omega^1}{dt} - (B-C)\,\omega^2\omega^3 &= mg\,(\beta\zeta_C - \gamma\eta_C)\,, \\ B\frac{d\omega^2}{dt} - (C-A)\,\omega^3\omega^1 &= mg\,(\gamma\xi_C - \alpha\zeta_C)\,, \\ C\frac{d\omega^3}{dt} - (A-B)\,\omega^1\omega^2 &= mg\,(\alpha\eta_C - \beta\xi_C)\,. \end{aligned} \tag{2}$$

This set of equations has to be adjoined with equations which describe time evolution of direction cosines of the unit vector $\mathbf{k}$.. These equations, known as Poisson's equations, arise from the condition $d\mathbf{k}/dt = 0$

$$\begin{aligned} \frac{d\alpha}{dt} &= \beta\omega^3 - \gamma\omega^2, \\ \frac{d\beta}{dt} &= \gamma\omega^1 - \alpha\omega^3, \\ \frac{d\gamma}{dt} &= \alpha\omega^2 - \beta\omega^1. \end{aligned} \tag{3}$$

In the literature (see, for example [8], [2]) system (3)-(4) is usually referred to as Euler-Poisson's equations.

System (3)-(4) consists of six ordinary differential equations of the first order. In general case, integrability of such a system can be afforded if there exist six independent first integrals (conservation laws, constants of motion). Their existence enables reduction of the problem to quadratures. Euler-Poisson's equations possess three independent first integrals

$$E = \frac{1}{2}\left[A\left(\omega^1\right)^2 + B\left(\omega^2\right)^2 + C\left(\omega^3\right)^2\right] + mg\left(\xi_C\alpha + \eta_C\beta + \zeta_C\gamma\right) = \text{const.}, \tag{4}$$

$$L_z = A\omega^1\alpha + B\omega^2\beta + C\omega^3\gamma = \text{const.}, \tag{5}$$

$$k^2 = \alpha^2 + \beta^2 + \gamma^2 = 1. \tag{6}$$

Equation (4) represents energy integral, (5) is angular momentum integral for the vertical Oz axis, while (6) represents well-known geometric relation. Since the system of Euler-Poisson's equations is autonomous, its order could be reduced by one. Furthermore, by means of Jacobi's theory of last multiplier (see [17], §119) it can be shown that it is sufficient to find only one additional first integral in order to prove complete integrability of the system (3)-(4). This conservation law does not exist in general case, i.e., for arbitrary values of parameters A, B, C, ξ_C, η_C and ζ_C, and huge efforts have been made towards dynamical specification of the body which permits the existence of additional first integral.

In its historical perspective, analysis of the motion of heavy top could be regarded as a history of the search for conservation laws. In the first period, which lasted almost 150 years, investigations were concentrated on the problem of finding completely integrable cases. Proof that there exists only a finite number of them turned attention to particular cases of integrability. In these cases integrability is afforded by the use of first integrals of particular character, where the constant of integration has value which is specified in advance. Obviously, these conservation laws impose constraints to the initial conditions of the problem. Main purpose of this paper is to analyze integrable cases of the motion of heavy top using one of the most powerful tools for obtaining conservation laws - theorem of Emmy Noether, which relates the existence of first integrals to invariance properties of Hamilton's action integral. The nature of the method will enable us to give geometrical explanation of symmetry transformations corresponding to certain conservation laws. The status of the first integrals of particular character within the framework of Noetherian approach will also be discussed.

2. *Integrable cases*

In this section we shall give a list of integrable cases which will be discussed in detail in the forthcoming analysis. Beside classical ones (Euler's, Lagrange's, and Kovalevskaya's), several particular cases of integrability will also be tackled. Their thorough study by means of Noether's theorem will be performed in section 5.

Euler's case. This case of integrability, obtained by Euler in 1750, describes inertial motion of heavy top since the center of mass coincides with a fixed point ($\xi_C = \eta_C = \zeta_C = 0$). In that case angular momentum vector is constant

$$\mathbf{L}_O = \text{const.}, \tag{7}$$

and Euler's dynamical equations have the following scalar conservation law

$$L_O^2 = A^2 \left(\omega^1\right)^2 + B^2 \left(\omega^2\right)^2 + C^2 \left(\omega^3\right)^2 = \text{const.} \tag{8}$$

Lagrange's case. In Lagrange's case (1788) it is assumed that rigid body possesses dynamical symmetry, meaning that center of mass lies on the axis of symmetry of the body: $A = B \neq C$, $\xi_C = \eta_C = 0$, $\zeta_C \neq 0$. In that case additional first integral, which evidently follows from Euler's dynamical equations, reads

$$\omega^3 = \text{const.} \tag{9}$$

General solution of the problem in the cases of Euler and Lagrange could be expressed in terms of elliptic functions.

Kovalevskaya's case. Over a century after Lagrange, Sophia Kovalevskaya (1889) derived third integrable case of the motion of heavy top.

Dynamical characteristics of the body describe so-called dynamically asymmetrical top, whose center of mass lies in the plane of equal moments of inertia: $A = B = 2C$, $\xi_C > 0$, $\eta_C = \zeta_C = 0$. If these conditions are fulfilled, system (3)-(4) has the following additional first integral

$$\left[\left(\omega^1\right)^2 - \left(\omega^2\right)^2 - \mu\alpha\right]^2 + \left(2\omega^1\omega^2 - \mu\beta\right)^2 = \text{const.}; \; \mu = \frac{mg\xi_C}{C}. \quad (10)$$

In this case general solution could be expressed in terms of hyperelliptic functions.

At this stage a few remarks should be given. The significance of the work of Sophia Kovalevskaya cannot be overestimated even today. It has a deep influence on contemporary investigations of integrable dynamical systems and conservation laws. It was for the first time that integrability of dynamical system was proved due to existence of the first integral other than classical ones[1] - Kovalevskaya's integral was of the fourth degree with respect to the components of angular velocity. Actually, this result opened a new chapter in the study of conservation laws by pointing out that first integrals do not always have an apparent physical meaning. In a certain sense, this was a logical consequence of the fact that Kovalevskaya's investigations were based upon purely mathematical considerations. Namely, she treated time as complex variable, and grounded her analysis on the hypothesis that singularities of the solution in complex plane could give suitable information about behavior of real solutions. This idea, in conjunction with Painlevé's analysis of singular points (see [5]), gave rise to a conjecture of Ablowitz et al. [1] which relates complete integrability of nonlinear evolution equations to ordinary differential equations without movable critical points. Finally, investigations of Kovalevskaya, and ones of Lyapunov and Poincaré shortly afterwards, proved that there are no other integrable cases of the motion of heavy top in classical sense, i.e., for arbitrary values of initial conditions. One of the deepest results concerning nonexistence of additional first integral is presented in the paper of Ziglin [18]. This fact turned attention to particular cases of integrability: in order to make system (3)-(4) integrable, conservation laws of particular character are employed. In this paper we shall discuss three such cases.

Hess-Appel'rot's case. This case was discovered independently by Hess (1890) and Appel'rot (1892). It is also called a case of a loxodromic

[1]By classical conservation laws we consider integrals of linear and angular momentum, and integral of energy.

pendulum. In this case it is supposed that center of mass lies on the line perpendicular to a plane whose intersection with reciprocal momental ellipsoid of inertia $(x^2/A + y^2/B + z^2/C = 1)$ is circular. This condition could be analytically expressed as follows:

$$\eta_C = 0;\ A\,(B - C)\,\xi_C^2 - C\,(A - B)\,\zeta_C^2 = 0. \tag{11}$$

Hess showed that if the angular momentum vector $\mathbf{L}_O$ is orthogonal to the position vector $\mathbf{r}_C$ at the initial instant of time, i.e., $\mathbf{L}_O \cdot \mathbf{r}_C = 0$, then the following first integral exists:

$$A\omega^1\xi_C + C\omega^3\zeta_C = 0. \tag{12}$$

In other words, condition of orthogonality remains fulfilled during the motion.

Case of Goryachev and Chaplyigin. In this case, discovered in 1900, body is dynamically asymmetric and center of mass lies in the plane of equal moments of inertia ($A = B = 4C$, $\xi_C > 0$, $\eta_C = \zeta_C = 0$). It is shown that if the angular momentum vector during the motion lies in the horizontal plane

$$L_z = A\omega^1\alpha + B\omega^2\beta + C\omega^3\gamma = 0, \tag{13}$$

then Euler-Poisson's equations possess the first integral which is of the third degree with respect to the components of angular velocity

$$\left(\left(\omega^1\right)^2 + \left(\omega^2\right)^2\right)\omega^3 - \mu\omega^1\gamma = \text{const.};\ \mu = \frac{mg\xi_C}{C}. \tag{14}$$

Grioli's case. This case was discovered in 1950, and it describes regular precession of a heavy top about nonvertical axis. Under the assumption that center of mass lies on the line perpendicular to the plane whose intersection with momental ellipsoid of inertia is circular, which could be written as

$$\xi_C\sqrt{B - C} - \zeta_C\sqrt{A - B} = 0 \ \Leftrightarrow\ \frac{A - B}{\xi_C^2} = \frac{B - C}{\zeta_C^2} = \frac{A - C}{\xi_C^2 + \zeta_C^2}, \tag{15}$$

along with $A > B > C$, and $\eta_C = 0$, Euler-Poisson's equations have the following first integral

$$\xi_C\omega^1 + \zeta_C\omega^3 = \kappa = \text{const.}, \tag{16}$$

adjoined with condition (particular first integral)

$$\left(A\xi_C\omega^1 + C\zeta_C\omega^3\right)\omega^2 - mgl^2\beta = 0;\ l^2 = \xi_C^2 + \zeta_C^2. \tag{17}$$

If the body performs regular precession, then the following relation is supposed to hold

$$\left(\omega^1\right)^2 + \left(\omega^2\right)^2 + \left(\omega^3\right)^2 = \text{const.} \tag{18}$$

It can be shown that this conservation law arises in conjunction with particular first integral of the form

$$B\left(\zeta_C\omega^1 - \xi_C\omega^3\right)\kappa - (A - C)\, l^2\omega^1\omega^3 + (\xi_C\gamma - \zeta_C\alpha)\, l^2 = 0, \tag{19}$$

where κ is defined by (16).

Common property of all above mentioned particular cases is that complete integrability is afforded by constructing a pair of conservation laws. One of them has a general character, while the other one is a particular first integral whose constant of integration has specified value, and thus imposes an additional constraint to the initial conditions of the motion.

3. *Noether's theorem in quasi-coordinates*

Quest for integrable cases is not the only aspect of the analysis of the motion of heavy top. For example, stability of steady motions could also be a very intriguing question (see [9], [4]). A lot of efforts have also been made towards discussion of known results using new methods. Beautiful examples of qualitative analysis could be found in works of Kozlov [6] and Arnol'd [3]. Another line of investigations is concerned with re-deriving integrable cases and their generalization by using the method of Lax pairs ([14], [11], [13], [15]). Present study is concerned with analysis of conservation laws for the motion of heavy top by means of Noether's theorem.

Let us consider Lagrangian dynamical system, whose state at any instant of time t is determined by n-tuple of generalized coordinates $\mathbf{x}(t) = \left(x^1(t), ..., x^n(t)\right)$, and n-tuple of generalized velocities $d\mathbf{x}(t)/dt = \left(dx^1(t)/dt, ..., dx^n()/dt\right)$. Its behavior could be completely described by a single function, the so-called Lagrangian function $L(t, \mathbf{x}, d\mathbf{x}/dt)$, and differential equations of motion have the structure of Euler-Lagrange's equation. Theorem of Emmy Noether, in its classical version, relates the existence of conservation laws of Lagrangian dynamical systems to invariance properties of Hamilton's action integral (for reference see [16]). As far as author is concerned, there have been only few attempts to study integrable cases of the motion of heavy top using Noether's theorem. One of the main problems in this approach arises when one wants to recast Euler-Poisson's equations into variational setup. The problem is not trivial, and it could be overcome in several ways. In present study we shall use a version of Noether's theorem formulated by Djukic [4]. Main feature of his approach lies in the fact that the state of the system is described by generalized coordinates and linear forms of generalized velocities, the so-called quasi-velocities. In this section we shall briefly summarize the results which will be used in forthcoming analysis. Summation convention with respect to repeated indices will be assumed. All functions that appear in the analysis are supposed to be sufficiently smooth.

Quasi-velocities are usually defined by the following relations

$$\omega^i = \alpha^i_j(\mathbf{x})\frac{dx^j}{dt}, \tag{20}$$

where functions $\alpha^i_j(\mathbf{x})$ are supposed to be continuous and have continuous first derivatives with respect to all variables in the region in which motion occurs.[2] Main characteristic of genuine quasi-velocities is non-integrability of equations (20). If they were integrable, one could derive them from relations which describe the change of coordinates. Otherwise, we could only formally define the so-called differentials of quasi-coordinates as linear differential forms

$$d\pi^i = \omega^i dt = \alpha^i_j(\mathbf{x})\, dx^j. \tag{21}$$

If we assume $\det\left[\alpha^i_j(\mathbf{x})\right] \neq 0$, we could obtain following relations from (20)

$$\frac{dx^i}{dt} = \beta^i_j(\mathbf{x})\,\omega^j. \tag{22}$$

Functions $\beta^i_j(\mathbf{x})$ are related to $\alpha^i_j(\mathbf{x})$ by obvious relations $\alpha^i_k\beta^k_j = \beta^i_k\alpha^k_j = \delta^i_j$, where δ^i_j is Kronecker's delta. According to this equation and equation (21) we could formally define differentiation with respect to quasi-coordinates

$$\begin{aligned} \frac{\partial}{\partial\pi^i} &= \beta^j_i(\mathbf{x})\frac{\partial}{\partial x^j}; \\ \frac{d}{dt} &= \frac{\partial}{\partial t} + \omega^i\frac{\partial}{\partial\pi^i}. \end{aligned} \tag{23}$$

[2]For a more general definition of quasi-velocities, which includes non-homogeniety and time-dependence of coefficients, see [16].

Finally, let us define the so-called objects of non-holonomity by following relations

$$\gamma^i_{jk} = \left(\frac{\partial \alpha^i_r}{\partial x^s} - \frac{\partial \alpha^i_s}{\partial x^r} \right) \beta^s_j \beta^r_k = \alpha^i_r \left(\frac{\partial \beta^r_j}{\partial x^s} \beta^s_k - \frac{\partial \beta^r_k}{\partial x^s} \beta^s_j \right),$$

with obvious skew-symmetric property $\gamma^i_{kj} = -\gamma^i_{jk}$.

We shall now assume that the state of dynamical system is determined by n-tuple of generalized coordinates and n-tuple of quasi-velocities $\boldsymbol{\omega}\ (t) = (\omega^1(t), ..., \omega^n(t))$. Lagrangian function of dynamical system could be transformed into a function of these state variables by the use of equation (22), i.e., $L = L(t, \mathbf{x}, \boldsymbol{\omega}\ )$. These assumptions lead to differential equations of motion in the form of Hamel-Boltzmann's equations

$$\frac{d}{dt}\frac{\partial L}{\partial \omega^i} + \gamma^k_{ji} \frac{\partial L}{\partial \omega^k} \omega^j - \frac{\partial L}{\partial \pi^i} = 0. \tag{24}$$

It could be shown that these equations arise as necessary conditions of extremum of Hamilton's action integral (see [10])

$$J = \int_{t_0}^{t_1} L(t, \mathbf{x}(t), \boldsymbol{\omega}\ (t))\, dt. \tag{25}$$

Let us turn our attention to invariance properties of the action integral (25). In order to analyze this question we shall introduce one-parameter infinitesimal transformations of time t and generalized coordinates $\mathbf{x}$ to another set of independent and dependent variables, such that each curve $t \to \mathbf{x}(t)$ is transformed to a parameter dependent curve $T \to \mathbf{X}(T)$ for sufficiently small value of parameter ε. In present study we shall use transformations of the form

$$\begin{aligned} T &= t + \varepsilon \tau (t, \mathbf{x}, \boldsymbol{\omega}\ ) + O\left(\varepsilon^2\right), \\ X^i &= x^i + \varepsilon \xi^i (t, \mathbf{x}, \boldsymbol{\omega}\ ) + O\left(\varepsilon^2\right), \end{aligned} \tag{26}$$

where τ and ξ^i are generators of transformations. Because of their implicit dependence on generalized velocities, these transformations fall into a category of Lie-Bäcklund tangent transformations or generalized symmetries [12]. Although, in general case, they cannot be put into a framework of classical Lie groups, they have a great practical significance. They enable us to establish one-to-one correspondence between conservation laws and infinitesimal transformations. For our analysis, it will be convenient to employ generators of coordinate transformations in the form

$$\xi^i (t, \mathbf{x}, \boldsymbol{\omega}) = \beta^i_j (\mathbf{x}) \psi^j (t, \mathbf{x}, \boldsymbol{\omega}). \tag{27}$$

Since transformations (26)-(27) map a curve to a family of curves, they imply apropriate transformations of generalized velocities

$$\frac{dX^i}{dT} = \frac{dx^i}{dt} + \varepsilon \left(\frac{d\xi^i}{dt} - \frac{dx^i}{dt}\frac{d\tau}{dt} \right) + O\left(\varepsilon^2\right). \tag{28}$$

Moreover, since defining equations of quasi-velocities (20) have to retain their form after transformations, i.e.,

$$\Omega^i = \alpha^i_j (\mathbf{X}) \frac{dX^i}{dT},$$

their infinitesimal transformations are also defined by (26)-(27)

$$\Omega^i = \omega^i + \varepsilon \left(\frac{d\psi^i}{dt} - \gamma^i_{jk} \omega^j \psi^k - \omega^i \frac{d\tau}{dt} \right) + O\left(\varepsilon^2\right). \tag{29}$$

Infinitesimal transformations (26)-(27) are called symmetry transformations of action integral (25) if they leave it invariant up to a gauge term, i.e., if there exists a gauge function $f(t, \mathbf{x}, \boldsymbol{\omega})$ such that for each differentiable curve $t \to \mathbf{x}(t)$ the following relation holds

$$\begin{aligned} \int_{T_0}^{T_1} L(T, \mathbf{X}(T), \boldsymbol{\Omega}(T))\, dT &= \int_{t_0}^{t_1} L(t, \mathbf{x}(t), \boldsymbol{\omega}\ (t))\, dt \\ &+\varepsilon \int_{t_0}^{t_1} \frac{df}{dt} (t, \mathbf{x}(t), \boldsymbol{\omega}\ (t))\, dt + O\left(\varepsilon^2\right). \end{aligned}$$

Action integral is said to be absolutely invariant if $f \equiv 0$. This definition could be transformed into a condition of invariance in the form

$$L\frac{d\tau}{dt} + \frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial \pi^i}\psi^i + \frac{\partial L}{\partial \omega^i} \left(\frac{d\psi^i}{dt} - \gamma^i_{jk} \omega^j \psi^k - \omega^i \frac{d\tau}{dt} \right) = \frac{df}{dt}. \tag{30}$$

By a straightforward calculation this relation could be reduced to a conservation law in characteristic form

$$\frac{d}{dt}\left[L\tau + \frac{\partial L}{\partial \omega^i} \left(\psi^i - \omega^i \tau \right) - f \right] = \left(\psi^i - \omega^i \tau \right) \left[\frac{d}{dt}\frac{\partial L}{\partial \omega^i} + \gamma^k_{ji} \frac{\partial L}{\partial \omega^k} \omega^j - \frac{\partial L}{\partial \pi^i} \right].$$

Thus, we can formulate Noether's theorem for Lagrangian systems in quasicoordinates: for every infinitesimal transformation of time and generalized coordinates which leaves Hamilton's action integral (25) invariant up to a gauge term, there corresponds conservation law of Lagrangian dynamical system described by Hamel-Boltzman's equations of the form

$$L\tau + \frac{\partial L}{\partial \omega^i}\left(\psi^i - \omega^i \tau\right) - f = \text{const.} \tag{31}$$

Our final remark is concerned with the fact that total time-derivatives of generators and gauge function contain derivatives of quasi-velocities. We shall assume that regularity condition is fulfilled ($\det\left[\partial^2 L/\partial\omega^i\partial\omega^j\right] \neq 0$), and that equations (24) could be solved with respect to derivatives of quasi-velocities, so that total time-derivatives will be calculated along trajectory of dynamical system.

4. *Application to the motion of heavy top*

In order to analyze conservation laws for the motion of heavy top by means of Noether's theorem we shall use Euler's angles $\psi = x^1$, $\theta = x^2$ and $\varphi = x^3$ as generalized coordinates. Quasi-velocities will be chosen as components of angular velocity with respect to a moving frame. Their defining equations are as follows

$$\begin{aligned}
\omega^1 &= \frac{dx^1}{dt}\sin x^2 \sin x^3 + \frac{dx^2}{dt}\cos x^3, \\
\omega^2 &= \frac{dx^1}{dt}\sin x^2 \cos x^3 - \frac{dx^2}{dt}\sin x^3, \\
\omega^3 &= \frac{dx^1}{dt}\cos x^2 + \frac{dx^3}{dt}.
\end{aligned} \tag{32}$$

These equations could be easily solved with respect to generalized velocities

$$\begin{aligned}
\frac{dx^1}{dt} &= \omega^1\frac{\sin x^3}{\sin x^2} + \omega^2\frac{\cos x^3}{\sin x^2}, \\
\frac{dx^2}{dt} &= \omega^1 \cos x^3 - \omega^2 \sin x^3, \\
\frac{dx^3}{dt} &= -\omega^1\frac{\cos x^2}{\sin x^2}\sin x^3 - \omega^2\frac{\cos x^2}{\sin x^2}\cos x^3 + \omega^3.
\end{aligned} \tag{33}$$

Since unit vector **k** of vertical Oz axis could be expressed in terms of Euler's angles

$$\mathbf{k} = \sin x^2 \sin x^3 \mathbf{e}_1 + \sin x^2 \cos x^3 \mathbf{e}_2 + \cos x^2 \mathbf{e}_3,$$

it is easy to show that Euler's dynamical equations (3) have the structure of Hamel-Boltzmann's equations for Lagrangian function in the form of kinetic potential (see [10], p.368, and [17], p.43)

$$\begin{aligned} L &= \frac{1}{2}\left[A\left(\omega^1\right)^2 + B\left(\omega^2\right)^2 + C\left(\omega^3\right)^2\right] \\ &\quad -mg\left(\xi_C \sin x^2 \sin x^3 + \eta_C \sin x^2 \cos x^3 + \zeta_C \cos x^2\right). \end{aligned} \qquad (34)$$

With objects of non-holonomity of the form $\gamma^1_{32} = \gamma^2_{13} = \gamma^3_{21} = 1$, $\gamma^1_{23} = \gamma^2_{31} = \gamma^3_{12} = -1$ equations (24) reduce to

$$\begin{aligned} A\frac{d\omega^1}{dt} - (B-C)\,\omega^2\omega^3 - mg\left(\zeta_C \sin x^2 \cos x^3 - \eta_C \cos x^2\right) &= 0, \\ B\frac{d\omega^2}{dt} - (C-A)\,\omega^3\omega^1 - mg\left(\xi_C \cos x^2 - \zeta_C \sin x^2 \sin x^3\right) &= 0, \\ C\frac{d\omega^3}{dt} - (A-B)\,\omega^1\omega^2 - mg\left(\eta_C \sin x^2 \sin x^3 - \xi_C \sin x^2 \cos x^3\right) &= 0. \end{aligned} \qquad (35)$$

Poisson's equations (4) are replaced by equations (33). It is interesting to note that generalized coordinate x^1 neither figures in equations (35), nor in (33). Moreover, by introduction of Euler's angles geometric first integral (6) reduces to identity.

Since our analysis tackles only time-independent first integrals, we shall confine ourselves with generators of transformations and gauge function which are also time-independent

$$\tau = \tau\left(\mathbf{x}, \boldsymbol{\omega}\right); \; \xi^i = \xi^i\left(\mathbf{x}, \boldsymbol{\omega}\right) = \beta^i_j\left(\mathbf{x}\right)\psi^j\left(\mathbf{x}, \boldsymbol{\omega}\right); \; f = f\left(\mathbf{x}, \boldsymbol{\omega}\right).$$

Intention of this paper is to give, if possible, a geometric interpretation of transformations which correspond to certain first integrals. In order to achieve this goal we shall define a vector of rotation as follows

$$\boldsymbol{\Psi} = \xi^1\left(\mathbf{x}, \boldsymbol{\omega}\right)\mathbf{k} + \xi^2\left(\mathbf{x}, \boldsymbol{\omega}\right)\mathbf{n} + \xi^3\left(\mathbf{x}, \boldsymbol{\omega}\right)\mathbf{e}_3, \qquad (36)$$

where **n** denotes unit vector of the line of nodes - intersection of the planes Oxy and $O\xi\eta$. For it will be useful to express rotation vector in fixed or moving frame, the following relations will be used in the forthcoming analysis

$$\begin{aligned} \mathbf{k} &= \sin x^2 \sin x^3 \mathbf{e}_1 + \sin x^2 \cos x^3 \mathbf{e}_2 + \cos x^2 \mathbf{e}_3, \\ \mathbf{n} &= \cos x^1 \mathbf{i} + \sin x^1 \mathbf{j} = \cos x^3 \mathbf{e}_1 - \sin x^3 \mathbf{e}_2, \\ \mathbf{e}_3 &= \sin x^1 \sin x^2 \mathbf{i} - \cos x^1 \sin x^2 \mathbf{j} + \cos x^2 \mathbf{k}. \end{aligned} \tag{37}$$

In the remainder of this section we shall give an outline of the method which will be used for derivation of conservation laws. According to traditional classification, we shall discuss first integrals having in mind the highest degree of quasi-velocities which appear in them.

Linear first integrals. In order to derive conservation laws linear with respect to quasi-velocities, we shall introduce generators of transformations in the form

$$\tau \equiv 0;\ \psi^i = \psi^i(\mathbf{x}),$$

and suppose that action integral is absolutely invariant, i.e., $f \equiv 0$. With these assumptions conservation law (31) have the form

$$A\psi^1(\mathbf{x})\,\omega^1 + B\psi^2(\mathbf{x})\,\omega^2 + C\psi^3(\mathbf{x})\,\omega^3 = \text{const.}, \tag{38}$$

while condition of invariance (30) reduces to a quadratic form of quasi-velocities

$$\begin{aligned} & A\left[\frac{\sin x^3}{\sin x^2}\frac{\partial\psi^1}{\partial x^1} + \cos x^3\frac{\partial\psi^1}{\partial x^2} - \frac{\cos x^2}{\sin x^2}\sin x^3\frac{\partial\psi^1}{\partial x^3}\right]\left(\omega^1\right)^2 + \\ & +B\left[\frac{\cos x^3}{\sin x^2}\frac{\partial\psi^2}{\partial x^1} - \sin x^3\frac{\partial\psi^2}{\partial x^2} - \frac{\cos x^2}{\sin x^2}\cos x^3\frac{\partial\psi^2}{\partial x^3}\right]\left(\omega^2\right)^2 + C\frac{\partial\psi^3}{\partial x^3}\left(\omega^3\right)^2 \\ & +\left\{A\left[\frac{\cos x^3}{\sin x^2}\frac{\partial\psi^1}{\partial x^1} - \sin x^3\frac{\partial\psi^1}{\partial x^2} - \frac{\cos x^2}{\sin x^2}\cos x^3\frac{\partial\psi^1}{\partial x^3}\right]\right. \\ & \left. +B\left[\frac{\sin x^3}{\sin x^2}\frac{\partial\psi^2}{\partial x^1} + \cos x^3\frac{\partial\psi^2}{\partial x^2} - \frac{\cos x^2}{\sin x^2}\sin x^3\frac{\partial\psi^2}{\partial x^3}\right] + (A-B)\,\psi^3\right\}\omega^1\omega^2 \\ & +\left\{C\left[\frac{\cos x^3}{\sin x^2}\frac{\partial\psi^3}{\partial x^1} - \sin x^3\frac{\partial\psi^3}{\partial x^2} - \frac{\cos x^2}{\sin x^2}\cos x^3\frac{\partial\psi^3}{\partial x^3}\right]\right. \\ & \left. +B\frac{\partial\psi^2}{\partial x^3} + (B-C)\,\psi^1\right\}\omega^2\omega^3 \\ & +\left\{C\left[\frac{\sin x^3}{\sin x^2}\frac{\partial\psi^3}{\partial x^1} + \cos x^3\frac{\partial\psi^3}{\partial x^2} - \frac{\cos x^2}{\sin x^2}\sin x^3\frac{\partial\psi^3}{\partial x^3}\right]\right. \end{aligned} \tag{39}$$

$$+A\frac{\partial\psi^1}{\partial x^3}+(C-A)\,\psi^2\Big\}\,\omega^3\omega^1$$

$$-mg\left[\left(\eta_C\cos x^2-\zeta_C\sin x^2\cos x^3\right)\psi^1+\left(\zeta_C\sin x^2\sin x^3-\xi_C\cos x^2\right)\psi^2\right.$$

$$\left.+\left(\xi_C\sin x^2\cos x^3-\eta_C\sin x^2\sin x^3\right)\psi^3\right]=0$$

As it was mentioned earlier, along with generators of transformations we have to adjust dynamical characteristics (parameters) of the body if conservation law is to be derived. This fact reveals the complexity of the problem. It will be treated in two different ways.

First way of analysis, originally proposed by Vujanovic [16], is based upon fact that condition of invariance represents polynomial form of quasi-velocities, with coefficients which are the functions of generalized coordinates solely. Thus, it could be decomposed into a set of linear partial differential equations of the first order, known as generalized Killing's equations, which serve for determination of generators of transformations and gauge function. By finding any non-trivial solution of this system we shall satisfy the condition of invariance identically, and produce a conservation law.

Second way of analysis is motivated by the presence of integrable cases which contain first integrals of particular character. In this approach we shall not decompose condition of invariance, but rather treat it as a single equation. Then, we shall propose generators of transformations and gauge function in such a way that certain relation between state variables holds as a consequence of equation (39). According to adopted terminology, this relation will represent a conservation law of particular character. These symmetry transformations will also produce a first integral of general character, which follows from equation (38). Thus, second way of analysis gives us a pair of conservation laws, particular and general, in such a way that particular one comes directly from the condition of invariance. As far as author is aware, this kind of treatment within Noetherian approach is not known in the literature.

Quadratic first integrals. In order to derive quadratic conservation laws, we have to use generators of transformations in the form

$$\tau=\tau\left(\mathbf{x}\right);\ \psi^\iota=a^i_j\left(\mathbf{x}\right)\omega^j,$$

while gauge function could be either function of generalized coordinates solely, or quadratic form of quasi-velocities. In either case, general form of conservation law is as follows:

$$
\begin{aligned}
&A\left[a_1^1(\mathbf{x})-\frac{1}{2}\tau(\mathbf{x})\right]\left(\omega^1\right)^2+B\left[a_2^2(\mathbf{x})-\frac{1}{2}\tau(\mathbf{x})\right]\left(\omega^2\right)^2\\
&+C\left[a_3^3(\mathbf{x})-\frac{1}{2}\tau(\mathbf{x})\right]\left(\omega^3\right)^2+\left[Aa_2^1(\mathbf{x})+Ba_1^2(\mathbf{x})\right]\omega^1\omega^2\\
&+\left[Ba_3^2(\mathbf{x})+Ca_2^3(\mathbf{x})\right]\omega^2\omega^3+\left[Ca_1^3(\mathbf{x})+Aa_3^1(\mathbf{x})\right]\omega^3\omega^1 \qquad (40)
\end{aligned}
$$

$$-\left\{mg\left[\xi_C\sin x^2\sin x^3+\eta_C\sin x^2\cos x^3+\zeta_C\cos x^2\right]\tau(\mathbf{x})+f(\mathbf{x},\boldsymbol{\omega})\right\}=\text{const.}$$

Condition of invariance will not be presented explicitly because of its complexity.

First integrals of third and fourth degree. Complexity of general form of conservation laws and conditions of invariance grows rapidly as the highest degree of quasi-velocities grows. For this reason, we shall give only basic facts concerned with higher-degree first integrals. If we are seeking for the first integral of n-th degree, then generator of time transformation $\tau(\mathbf{x},\boldsymbol{\omega})$ has to have polynomial form of degree $n-2$ with respect to quasi-velocities, while generators of coordinate transformations $\psi^i(\mathbf{x},\omega)$ have to be polynomials of degree $n-1$. Degree of the gauge function does not have to exceed the degree of conservation law. Finally, procedures described for linear first integrals are valid in other cases also.

5. *Results*

In this section we shall re-derive conservation laws for the motion of heavy top quoted earlier in the text. Particular attention will be devoted to the possibility of geometrical interpretation of symmetry transformations, and to the analysis of the first integrals of particular character.

Integral of energy. In order to derive integral of energy, which is a quadratic one, we shall choose generators and gauge function in such a way that they describe time translation, i.e.,

$$\tau=K=\text{const.};\ \psi^\iota=0;\ f=0.$$

It is easy to check that condition of invariance (30) is satisfied identicaly for arbitrary values of dynamical parameters, and that energy integral follows directly from equation (40)

$$\frac{1}{2}\left[A\left(\omega^1\right)^2+B\left(\omega^2\right)^2+C\left(\omega^3\right)^2\right]$$
$$+mg\left(\xi_C\sin x^2\sin x^3+\eta_C\sin x^2\cos x^3+\zeta_C\cos x^2\right)=\text{const.}$$

It could also be shown that the same result comes out from the following choice of generators and gauge function

$$\tau=0;\ \psi^i=K\omega^i;\ K=\text{const.};$$
$$f=-2Kmg\left(\xi_C\sin x^2\sin x^3+\eta_C\sin x^2\cos x^3+\zeta_C\cos x^2\right).$$

Djukic [4] has shown that energy integral could be derived for the same choice of generators of transformations, but using a gauge function in the form of Lagrangian function $f=KL\left(\mathbf{x},\boldsymbol{\omega}\right)$. This result is in full agreement with results obtained for Lagrangian systems expressed in usual manner (see [16], p.93). First possibility reminds us that integral of energy is tightly connected to homogeneity of time in classical mechanics, while the other ones show that solution to particular problems may not be unique.

Angular momentum integral. This conservation law (5) is of the first degree with respect to quasi-velocities, and it is valid for arbitrary values of dynamical parameters. By applying the first way of analysis, described in previous section, we can find solution of generalized Killing's equations in the form

$$\psi^1=\sin x^2\sin x^3;\ \psi^2=\sin x^2\cos x^3;\ \psi^3=\cos x^2.$$

Then, from equation (38) there follows angular the momentum conservation law

$$A\omega^1\sin x^2\sin x^2+B\omega^2\sin x^2\cos x^3+C\omega^3\cos x^2=\text{const.}$$

Generators of transformations possess quite obvious geometrical interpretation based upon equations (36)-(37). It could be shown that vector of rotation reduces to $\boldsymbol{\Psi}=\mathbf{k}$, which implies that angular momentum integral comes out as a consequence of absolute invariance of the action integral with respect to infinitesimal rotation about vertical Oz axis. This property has already been known for Lagrangian systems expressed in terms of generalized coordinates and generalized velocities.

Euler's case. In the case of inertial motion of the rigid body analysis of quadratic first integrals reveals that condition of invariance (30) will be satisfied identically if action integral is absolutely invariant ($f = 0$) with respect to a coordinate transformation, whose generators are chosen to be components of angular mometum (see (2)) with respect to a moving frame

$$\psi^1 = A\omega^1;\ \psi^2 = B\omega^2;\ \psi^3 = C\omega^3,$$

while time variable does not suffer any transformation ($\tau = 0$). Then, from equation (40) there follows Euler's first integral

$$L_O^2 = A^2\left(\omega^1\right)^2 + B^2\left(\omega^2\right)^2 + C^2\left(\omega^3\right)^2 = \text{const.}$$

Having in mind vectorial form (7) of Euler's first integral, showing that angular momentum vector is fixed in space, we could give a very exhaustive geometrical analysis of symmetry transformations. Let us determine unit vector of angular momentum

$$\mathbf{e} = \frac{1}{L_O}\mathbf{L}_O = e_x\mathbf{i} + e_y\mathbf{j} + e_z\mathbf{k},$$

whose components with respect to a fixed frame are

$$\begin{aligned} e_x &= \frac{A\omega^1}{L_O}\left(\cos x^1 \cos x^3 - \sin x^1 \cos x^2 \sin x^3\right) \\ &+\frac{B\omega^2}{L_O}\left(-\cos x^1 \sin x^3 - \sin x^1 \cos x^2 \cos x^3\right) \\ &+\frac{C\omega^3}{L_O}\sin x^1 \sin x^2; \end{aligned}$$

$$\begin{aligned} e_y &= \frac{A\omega^1}{L_O}\left(\sin x^1 \cos x^3 + \cos x^1 \cos x^2 \sin x^3\right) \\ &+\frac{B\omega^2}{L_O}\left(-\sin x^1 \sin x^3 + \cos x^1 \cos x^2 \cos x^3\right) \\ &+\frac{C\omega^3}{L_O}\left(-\cos x^1 \sin x^2\right); \end{aligned}$$

$$e_z = \frac{A\omega^1}{L_O}\sin x^2 \sin x^3 + \frac{B\omega^2}{L_O}\sin x^2 \cos x^3 + \frac{C\omega^3}{L_O}\cos x^2.$$

By a straightforward calculation it could be shown that generators of transformation determine infinitesimal rotation about an axis which coincides with the vector of angular momentum

$$\boldsymbol{\Psi} = L_O\left(e_x\mathbf{i} + e_y\mathbf{j} + e_z\mathbf{k}\right) = L_O\mathbf{e} = \mathbf{L}_O.$$

Furthermore, vectorial first integral implies existence of three scalar conservaton laws

$$L_x = \mathbf{L}_O \cdot \mathbf{i} = \text{const.};\ L_y = \mathbf{L}_O \cdot \mathbf{j} = \text{const.};\ L_z = \mathbf{L}_O \cdot \mathbf{k}.$$

First of them could be obtained using generators of the form

$$\begin{aligned}
\psi^1 &= \mathbf{e}_1 \cdot \mathbf{i} = \cos x^1 \cos x^3 - \sin x^1 \cos x^2 \sin x^3; \\
\psi^2 &= \mathbf{e}_2 \cdot \mathbf{i} = -\cos x^1 \sin x^3 - \sin x^1 \cos x^2 \cos x^3; \\
\psi^3 &= \mathbf{e}_3 \cdot \mathbf{i} = \sin x^1 \sin x^2.
\end{aligned}$$

For derivation of the second one we have to use the following generators

$$\begin{aligned}
\psi^1 &= \mathbf{e}_1 \cdot \mathbf{j} = \sin x^1 \cos x^3 + \cos x^1 \cos x^2 \sin x^3; \\
\psi^2 &= \mathbf{e}_2 \cdot \mathbf{j} = -\sin x^1 \sin x^3 + \cos x^1 \cos x^2 \cos x^3; \\
\psi^3 &= \mathbf{e}_3 \cdot \mathbf{j} = -\cos x^1 \sin x^2.
\end{aligned}$$

Third conservation law is angular momentum integral for vertical axis derived above for arbitrary values of dynamical parameters.

Lagrange's case. In this case, conservation law (9) could be obtained as a consequence of absolute invariance of the action integral with respect to the following transformations

$$\tau = 0;\ \psi^1 = \psi^2 = 0;\ \psi^3 = K = \text{const.}$$

Their geometrical interpretation is almost inevitable - they describe infintesimal rotation about an axis of symmetry of the momental ellipsoid of inertia

$$\mathbf{\Psi} = \psi^3 \mathbf{e}_3 = K \mathbf{e}_3.$$

Kovalevskaya's case. Additonal first integral in the case of Sophia Kovalevskaya (10) is of the fourth degree with respect to quasi-velocities. Careful study of the condition of invariance (30) and general form of conservation law (31) revealed that Kovalevskaya's integral, which reads

$$\left[\left(\omega^1\right)^2 - \left(\omega^2\right)^2 - \mu \sin x^2 \sin x^3\right]^2 + \left(2\omega^1\omega^2 - \mu \sin x^2 \cos x^3\right)^2 = \text{const.},$$

where $\mu = mg\xi_C/C$, comes out by means of the following transformations

$$\tau = 0;$$

$$\begin{aligned}
\psi^1 &= \omega^1\left(\left(\omega^1\right)^2+\left(\omega^2\right)^2\right)-\mu\left(\omega^1\sin x^2\sin x^3+\omega^2\sin x^2\cos x^3\right);\\
\psi^2 &= \omega^2\left(\left(\omega^1\right)^2+\left(\omega^2\right)^2\right)-\mu\left(\omega^1\sin x^2\cos x^3-\omega^2\sin x^2\sin x^3\right);\\
\psi^3 &= 0,
\end{aligned}$$

along with the assumption of gauge-invariance for a gauge function

$$f=2C\mu\left[\left(\left(\omega^1\right)^2-\left(\omega^2\right)^2\right)\sin x^3+2\omega^1\omega^2\cos x^3-\mu\sin x^2\right]\sin x^2.$$

This solution is not unique. We could arrive to the same result using following transformations

$$\begin{aligned}
\psi^1 &= \psi^2=\psi^3=0;\\
\tau &= KL(\mathbf{x},\boldsymbol{\omega});\ K=\text{const.},
\end{aligned}$$

while gauge function is of the fourth degree with respect to quasi-velocities

$$\begin{aligned}
f =& -KC^2\left[\left(\left(\omega^1\right)^2+\left(\omega^2\right)^2+\frac{1}{4}\left(\omega^3\right)^2\right)\left(\omega^3\right)^2\right.\\
&+2\mu\left(\left(\omega^1\right)^2-\left(\omega^2\right)^2\right)\sin x^2\sin x^3+4\mu\omega^1\omega^2\sin x^2\cos x^3\\
&\left.-\mu^2\sin^2 x^2\left(1+\sin^2 x^3\right)\right].
\end{aligned}$$

Let us conclude the analysis of Kovalevskaya's case with remark that these symmetry transformations differs from the one obtained earlier by Djukic [4].

Hess-Appel'rot's case. In order to reconstruct Hess-Appel'rot's case let us suppose the following form of generators of transformations and gauge function

$$\tau=0;\ \psi^1=\text{const.};\ \psi^2=0;\ \psi^3=\text{const.};\ f=0.$$

This choice discloses our intention to construct linear conservation law of general character. Under these assumptions condition of invariance (39) reduces to

$$\psi^3(A-B)\,\omega^1\omega^2+\psi^1(B-C)\,\omega^2\omega^3-mg\left(\psi^3\xi_C-\psi^1\zeta_C\right)\sin x^2\cos x^3=0. \tag{41}$$

If we introduce further assumption

$$\psi^3\xi_C-\psi^1\zeta_C=0.$$

along with equivalent form of the condition (11), which reads

$$\frac{(B-C)\,\xi_C}{(A-B)\,\zeta_C}=\frac{C\zeta_C}{A\xi_C},$$

equation (41) will reduce to the following expression

$$\frac{\psi^3}{\xi_C}\frac{A-B}{A}\left(A\omega^1\xi_C+C\omega^3\zeta_C\right)\omega^2=0.$$

Condition of orthogonality of angular momentum vector to the position vector of the center of mass, which is particular first integral in Hess-Appel'rot's case, inevitably follows from this equation

$$A\omega^1\xi_C+C\omega^3\zeta_C=0.$$

Moreover, transformations determined in this way imply existence of general conservation law, which follows from equation (38)

$$A\omega^1\xi_C+C\omega^3\zeta_C=\text{const.}$$

It has to be pointed out that constant of integration has to be equal to zero because of the presence of the first integral of particular character. This case was discussed earlier by Langlois [7]. Finally, generators of transformations could be interpreted as rotation about an axis fixed in the body

$$\boldsymbol{\Psi}=\psi^1\mathbf{e}_1+\psi^3\mathbf{e}_3=\frac{\psi^1}{\xi_C}\mathbf{r}_C.$$

Case of Goryachev and Chaplyigin. Careful analysis of the general form of conservation law (31) and equation (14) revealed that generators of transformation have to be chosen in such a way that they describe coordinate transformation ($\tau=0$)

$$\psi^1=\frac{4}{5}\omega^1\omega^3-\frac{mg\xi_C}{C}\cos x^2;\ \psi^2=\frac{4}{5}\omega^2\omega^3;\ \psi^3=\frac{4}{5}\left(\left(\omega^1\right)^2+\left(\omega^2\right)^2\right).$$

If we assume absolute invariance of action integral ($f=0$), then condition of invariance (30) reduces to

$$\frac{mg\xi_C}{C}\left[4C\omega^1\sin x^2\sin x^3+4C\omega^2\sin x^2\cos x^3+C\omega^3\cos x^2\right]\omega^2=0.$$

This relation inevitably imposes a restriction to the angular momentum integral for vertical axis

$$4C\omega^1 \sin x^2 \sin x^3 + 4C\omega^2 \sin x^2 \cos x^3 + C\omega^3 \cos x^2 = 0,$$

which is the desired particular first integral. Along with this result, equation (31) generates additional first integral of general character - integral of Goryachev and Chaplyigin

$$\left(\left(\omega^1\right)^2 + \left(\omega^2\right)^2\right)\omega^3 - \frac{mg\xi_C}{C}\omega^1 \cos x^2 = \text{const.}$$

Grioli's case. Since Grioli's conservation law (16) is linear with respect to quasi-velocities, we shall assume the following form of generators and gauge function

$$\tau = 0;\ \psi^i = \text{const.};\ f = 0.$$

Condition of invariance (39) then reduces to th following form

$$\psi^3 (A - B)\,\omega^1\omega^2 + \psi^1 (B - C)\,\omega^2\omega^3 + \psi^2 (C - A)\,\omega^3\omega^1$$
$$-mg\left[\psi^2\left(\zeta_C \sin x^2 \sin x^3 - \xi_C \cos x^2\right)\right.$$
$$\left.+\psi^3\xi_C \sin x^2 \cos x^3 - \psi^1\zeta_C \sin x^2 \cos x^3\right] = 0.$$

If we choose generators of transformations as follows:

$$\psi^1 = \frac{C}{\zeta_C};\ \psi^2 = 0;\ \psi^3 = \frac{A}{\xi_C},$$

then by the use of relation (15) condition of of invariance could be easily reduced to relation

$$\frac{A - C}{l^2}\left[\left(A\omega^1\xi_C + C\omega^3\zeta_C\right)\omega^2 - mgl^2 \sin x^2 \cos x^3\right] = 0,$$

which trivially generates particular first integral

$$\left(A\omega^1\xi_C + C\omega^3\zeta_C\right)\omega^2 - mgl^2 \sin x^2 \cos x^3 = 0, \tag{42}$$

where $l^2 = \xi_C^2 + \zeta_C^2$. Under these assumptions, equation (38) reproduces Grioli's conservation law

$$\omega^1\xi_C + \omega^3\zeta_C = \kappa = \text{const.} \tag{43}$$

It could be noted that above defined generators determine infinitesimal rotation about an axis fixed in the body

$$\mathbf{\Psi} = \frac{AC}{\xi_C \zeta_C} \left(\frac{\xi_C}{A} \mathbf{e}_1 + \frac{\zeta_C}{C} \mathbf{e}_3 \right).$$

If the first integral of regular precession (18)

$$\left(\omega^1\right)^2 + \left(\omega^2\right)^2 + \left(\omega^3\right)^2 = \text{const.}$$

is to be derived by means of Noether's theorem, we have to use generators of transformations in the following form

$$\tau = 0;\ \psi^1 = \frac{\omega^1}{A};\ \psi^2 = \frac{\omega^2}{B};\ \psi^3 = \frac{\omega^3}{C};\ f = 0.$$

Then, using equations (42) and (43) we can derive the following relation from the condition of invariance (30)

$$\frac{\omega^2}{Bl^2} \Big[B\kappa \left(\omega^1 \zeta_C - \omega^3 \xi_C \right) - (A - C)\, l^2 \omega^1 \omega^3$$
$$+ l^2 \left(\xi_C \cos x^2 - \zeta_C \sin x^2 \sin x^3 \right) \Big] = 0,$$

which leads to a particular first integral for the regular precession about non-vertical axis

$$B\kappa \left(\omega^1 \zeta_C - \omega^3 \xi_C \right) - (A - C)\, l^2 \omega^1 \omega^3 + l^2 \left(\xi_C \cos x^2 - \zeta_C \sin x^2 \sin x^3 \right) = 0.$$

6. *Concluding remarks*

Although this study does not contain new integrable cases for the motion of heavy top, it sheds light on certain important aspects of the problem which arise in the context of Noether's theorem. Despite the fact that we relied our analysis on the Noether's theorem expressed in quasi-coordinates, symmetry transformations which lead to classical conservation laws (energy integral and angular momentum integral) retained properties known in the classical approach. This result leads to a conclusion that relation of conservation laws to the symmetry properties of space and time in classical mechanics is inherent part of dynamical system, independent on the choice of parameters which describe its behavior. Whenever it was possible, we have given geometrical interpretation of infinitesimal transformation, which completes the picture about symmetry analysis based upon Noether's theorem. Finally, for the first time there has been performed a thorough study of conservation laws of particular character from the Noetherian point of view. It was demonstrated that these first integrals colud be derived directly from the condition of invariance of the action integral, along with one general first integral which comes from the statement of Noether's theorem.

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