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A note on generalization of the Lewis invariant and the Ermakov systems

Srboljub S Simic

Faculty of Engineering, Department of Applied Mechanics, University of Novi Sad, Trg Dositeja Obradovica 6, 21000 Novi Sad, Yugoslavia

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Abstract. In this paper we examine several possible generalizations of the well known Lewis invariant and the Ermakov systems. The analysis relies on the application of the theorem of Emmy Noether. The problem of constructing Lewis-type invariants is reduced to a problem of solving a single partial differential equation—the so-called potential equation. Within this approach, the role of the auxiliary equation is clarified by an analysis of linear systems. Generalization of the Ermakov-type procedure for derivation of the Lewis invariant is also discussed. Moreover, different forms of general solution of the problem will be provided, which extend the range of possible applications of the analysis.

1. Introduction

Among the huge number of interesting and useful results concerning conservation laws, (first integrals, invariants and conserved quantities), the Lewis invariant (Lewis 1968) has attracted considerable attention. Namely, he found that the linear time-dependent harmonic oscillator

$$\ddot{x} + \omega^2(t)x = 0 \tag{1}$$

possesses an exact invariant of the form

$$I = [\rho(t)\dot{x} - \dot{\rho}(t)x]^2 + \left[\frac{x}{\rho(t)}\right]^2 = \text{constant} \tag{2}$$

where $\rho(t)$ is any solution of the auxiliary equation

$$\ddot{\rho}(t) + \omega^2(t)\rho(t) = \rho^{-3}(t). \tag{3}$$

At first glance it is clear that we are not dealing with a single invariant but with a whole class of invariants, since any particular solution of the auxiliary equation (3) leads to a first integral (2). It is also interesting to note that the dynamical equation (1) and the auxiliary equation (3) could exchange roles without affecting the general conclusion. That is, one could say that the dynamical system (3) possesses a conservation law described by equation (2) where $x(t)$ represents any solution of auxiliary equation (1). Using this fact, Ermakov (1880) derived the same invariant by eliminating the frequency $\omega^2(t)$. Inspired by Ermakov's method, several authors (Ray and Reid 1979a) have treated the problem in the same manner and attempted to construct certain pairs of coupled second-order nonlinear differential equations which are joined by the Lewis-type invariant. These classes of systems, which represent a generalization of the original system (1)–(3), are known as Ermakov systems. Detailed discussion of these

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systems, and of their outstanding property of nonlinear superposition, can be found in Sarlet and Ray (1981).

Here we shall analyse the Lewis invariant and the Ermakov systems within the context of the theorem of Noether (1918). We shall try to clarify the status of the auxiliary equation in this approach and obtain certain new results concerning linear non-autonomous systems by generalizing the Ermakov procedure. Our approach to the problem of constructing Lewis-type invariants for coupled nonlinear systems will be based on solving a single partial differential equation—the so-called potential equation. By deriving its general solution we shall extend the class of Ermakov systems, previously known in the literature (Ray and Reid 1979b, Kaushal and Korsch 1981), linked by generalized Lewis invariants. Most importantly, three different possible forms of Ermakov systems are presented. One of these is determined by an implicitly defined potential, and appears in this context for the first time.

2. Method

In the search for conservation laws, various methods can be applied. Group-theoretical methods, such as Noether's theorem and Lie's method of extended groups, have played an important role in the analysis of the Lewis invariant and the Ermakov systems. Much of the popularity of these methods is due to the fact that the conditions for infinitesimal invariance are linear. In this section we shall briefly present Noether's theorem for Lagrangian systems in a form which will enable us to analyse thoroughly the problems mentioned in the previous section. A detailed analysis of Noether's theorem and Lie groups can be found in Olver (1993) and Lovelock and Rund (1975). A simple proof of the theorem based on classical variational calculus, as well as many examples of its application to the various physical problems can be found in Vujanovic and Jones (1989).

Let us consider a dynamical system the state of which is determined by the n-tuple of generalized coordinates $\boldsymbol{x}(t) = (x^1(t), \ldots, x^n(t))$, and the n-tuple of generalized velocities $\dot{\boldsymbol{x}}(t) = (\dot{x}^1(t), \ldots, \dot{x}^n(t))$, where the overdot represents a derivative with respect to an independent variable, time t. Furthermore, let us suppose that there exists a single function, the so-called Lagrangian function $L(t, \boldsymbol{x}, \dot{\boldsymbol{x}})$, which serves as a complete description of the behaviour of the system, i.e. differential equations of motion possess the form of the Euler–Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = 0 \qquad i = 1, \ldots, n. \tag{4}$$

It is well known that these equations emerge as a necessary condition for the extremum of Hamilton's action integral

$$\delta \int_{t_0}^{t_1} L\big(t, \boldsymbol{x}(t), \dot{\boldsymbol{x}}(t)\big)\, \mathrm{d}t = 0$$

where t_0 and t_1 represent the initial and final values of time variables in which the position of the system is specified completely. With the usual assumption of the regularity of the Lagrangian ($\det(\partial^2 L/\partial \dot{x}^i \partial \dot{x}^j) \neq 0$), the Euler–Lagrange equations (4) can be recast into the normal form

$$\ddot{x}^i = F^i(t, \boldsymbol{x}, \dot{\boldsymbol{x}}) \qquad i = 1, \ldots, n. \tag{5}$$

Let us now consider an infinitesimal transformation of time t and generalized coordinates $\boldsymbol{x}$ to another set of independent and dependent variables $(t, \boldsymbol{x}) \to (T, \boldsymbol{X})$ such that each curve $t \to \boldsymbol{x}(t)$ is transformed into a parameter-dependent curve $T \to \boldsymbol{X}(T)$ for a sufficiently small value of the real parameter ϵ. At this stage, a question on the structure of the infinitesimal transformation can be posed. Indeed, this problem has been addressed by several authors, including Noether (1918). In the classical Lie-group treatment, generators of infinitesimal transformation are functions solely of time and generalized coordinates, and they act primarily as geometrical transformations. The introduction of velocities or even higher-order derivatives of generalized coordinates leads to a rather general type of transformation known as Lie–Bäcklund tangent transformations (Ibragimov and Anderson 1977) or generalized symmetries (Olver 1993). We shall confine ourselves to the following form of infinitesimal transformation:

$$T = t + \varepsilon\tau(t, \boldsymbol{x}, \dot{\boldsymbol{x}}) \qquad X^i = x^i + \varepsilon\xi^i(t, \boldsymbol{x}, \dot{\boldsymbol{x}}) \tag{6}$$

where τ and ξ^i are generators of transformation. This kind of transformation was introduced into the treatment of Noether's theorem by Djukic (1973). An infinitesimal transformation (6) is called a generalized variational symmetry if it leaves the action integral gauge-invariant (invariant up to a gauge term), meaning that there exists a gauge function $f(t, \boldsymbol{x}, \dot{\boldsymbol{x}})$ such that for each differentiable curve $t \to \boldsymbol{x}(t)$ the following relation holds:

$$\int_{T_0}^{T_1} L\left(T, \boldsymbol{X}(T), \frac{\mathrm{d}\boldsymbol{X}(T)}{\mathrm{d}T}\right) \mathrm{d}T = \int_{t_0}^{t_1} L(t, \boldsymbol{x}(t), \dot{\boldsymbol{x}}(t))\, \mathrm{d}t + \varepsilon \int_{t_0}^{t_1} \dot{f}(t, \boldsymbol{x}(t), \dot{\boldsymbol{x}}(t))\, \mathrm{d}t + \mathrm{O}(\varepsilon^2).$$

This definition is equivalent to the condition of invariance with the form

$$L\dot{\tau} + \frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial x^i}\xi^i + \frac{\partial L}{\partial \dot{x}^i}\left(\dot{\xi}^i - \dot{x}^i\dot{\tau}\right) = \dot{f} \tag{7}$$

where summation with respect to repeated indices is assumed. Moreover, a conservation law

$$L\tau + \frac{\partial L}{\partial \dot{x}^i}\left(\xi^i - \dot{x}^i\tau\right) - f = \text{constant} \tag{8}$$

of the dynamical system described by the Euler–Lagrange equations (4) corresponds to a generalized variational symmetry of an action integral. This is the statement of Noether's theorem in its classical form. It must be noted that in our analysis total derivatives with respect to time will be calculated along the trajectory of the dynamical system described in its normal form (5)

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{x}^i\frac{\partial}{\partial x^i} + F^i(t, \boldsymbol{x}, \dot{\boldsymbol{x}})\frac{\partial}{\partial \dot{x}^i}.$$

From the group-theoretical point of view, infinitesimal transformation (6) determines the vector field and its prolongation which also has to be calculated along the trajectory of the dynamical system. At the same time, the question of the determination of the one-parameter group action, as well as its prolongation, is rather cumbersome in the context of classical Lie-group theory. On the other hand, Noether's theorem, as presented here, remains a reliable tool in the search for conservation laws, since it provides a one-to-one correspondence between them and generalized variational symmetries. In this kind of study the condition of invariance (7), also known as the basic identity (Vujanovic and Jones 1989) and the Noether–Bessel–Hagen equation (Djukic and Vujanovic 1975), is of the utmost importance. It can serve for testing whether or not a proposed set of generators define a generalized variational symmetry, but it can also be used as a functional relation for their determination. The condition of invariance can also be decomposed into a set of first-order partial differential equations known as the generalized Killing equations (Vujanovic and Jones 1989). Any non-trivial solution of this system reveals the structure of the infinitesimal transformation and the gauge function, and also determines the structure of the conservation law (8). The solution of the system of generalized Killing equations has to be examined carefully since there are pathological situations in which a non-trivial solution leads to a trivial conservation law, i.e. a mathematical constant. The following discussion will rely on the use of the condition of invariance in order to determine the generators of infinitesimal transformation which will enable us to construct Lewis-type invariants and analyse possible generalizations of the Ermakov systems.

3. Quadratic invariants of one-dimensional systems

The study of the Lewis invariant usually starts as a search for a conservation law of the prescribed one-dimensional system. This present analysis will also be based on the one-dimensional approach. In order to achieve our goal we shall begin the study with a rather general assumption concerning the structure of a dynamical system. Namely, we shall suppose that the Lagrangian function possesses the traditional form of kinetic potential

$$L(t, x, \dot{x}) = \tfrac{1}{2}\dot{x}^2 - \Pi(t, x)$$

which leads to a single Euler–Lagrange differential equation

$$\ddot{x} = -\frac{\partial \Pi(t, x)}{\partial x}.$$

Careful inspection of equation (8) suggests the following form of generators and gauge function in order to obtain a conservation law which is quadratic with respect to generalized velocities:

$$\tau(t, x, \dot{x}) = \tau_0(t, x)$$
$$\xi(t, x, \dot{x}) = \xi_0(t, x) + \xi_1(t, x)\dot{x}$$
$$f(t, x, \dot{x}) = f_0(t, x) + f_1(t, x)\dot{x}.$$

Functions τ_0, ξ_0, ξ_1, f_0 and f_1 will be determined in the course of the analysis. This structure of generators has already been anticipated by Kobussen (1980) in the study of quadratic invariants of n-dimensional Lagrangian systems. According to these suppositions, the conservation law (8) reduces to the following quadratic invariant:

$$I = \left[\xi_1(t, x) - \tfrac{1}{2}\tau_0(t, x)\right]\dot{x}^2 + [\xi_0(t, x) - f_1(t, x)]\dot{x} - [\Pi(t, x)\tau_0(t, x) + f_0(t, x)] = \text{constant}. \tag{9}$$

The condition of invariance (7) then reduces to an expression which is polynomial with respect to the generalized velocity,

$$\left\{\frac{\partial}{\partial x}\left[\xi_1 - \tfrac{1}{2}\tau_0\right]\right\}\dot{x}^3 + \left\{\frac{\partial}{\partial t}\left[\xi_1 - \tfrac{1}{2}\tau_0\right] + \frac{\partial}{\partial x}[\xi_0 - f_1]\right\}\dot{x}^2 + \left\{\frac{\partial}{\partial t}[\xi_0 - f_1] - \frac{\partial}{\partial x}[\Pi\tau_0 + f_0] - 2\frac{\partial \Pi}{\partial x}\left[\xi_1 - \tfrac{1}{2}\tau_0\right]\right\}\dot{x} - \left\{\frac{\partial}{\partial t}[\Pi\tau_0 + f_0] + \frac{\partial \Pi}{\partial x}[\xi_0 - f_1]\right\} = 0.$$

One can easily see that the coefficients in the above expression do not depend on the generalized velocity. Thus, the equation will be satisfied identically if all coefficients vanish simultaneously. In other words, we can decompose the last equation into a set of first-order partial differential equations, the generalized Killing equations,

$$\frac{\partial}{\partial x}\left[\xi_1(t,x) - \tfrac{1}{2}\tau_0(t,x)\right] = 0 \tag{10}$$

$$\frac{\partial}{\partial t}\left[\xi_1(t,x) - \tfrac{1}{2}\tau_0(t,x)\right] + \frac{\partial}{\partial x}[\xi_0(t,x) - f_1(t,x)] = 0 \tag{11}$$

$$\frac{\partial}{\partial t}[\xi_0(t,x) - f_1(t,x)] - \frac{\partial}{\partial x}[\Pi(t,x)\tau_0(t,x) + f_0(t,x)] \\ -2\frac{\partial \Pi(t,x)}{\partial x}\left[\xi_1(t,x) - \tfrac{1}{2}\tau_0(t,x)\right] = 0 \tag{12}$$

$$\frac{\partial}{\partial t}[\Pi(t,x)\tau_0(t,x) + f_0(t,x)] + \frac{\partial \Pi(t,x)}{\partial x}[\xi_0(t,x) - f_1(t,x)] = 0. \tag{13}$$

At this point a few comments are in order. We have arrived at a system of four partial differential equations with six unknown functions $\tau_0, \xi_0, \xi_1, f_0, f_1$ and Π. At first sight, a very high degree of non-determination occurs in the problem. On the other hand, we must be aware of the fact that there are no other constraints, such as initial or boundary conditions, imposed on the system (10)–(13). Thus, we are seeking any non-trivial solution of the problem. Fortunately, there is a rather simple way of dealing with these difficulties. Since generalized Killing equations are adjusted according to the form of the coefficients in the expression for the conservation law (9), we can use the system (10)–(13) for their determination and for finding the form of the potential $\Pi(t,x)$ which allows the existence of the quadratic invariant. In the present analysis we shall proceed in the following way: equations (10)–(12) will be used for successive calculations of the coefficients, while equation (13) will serve to reveal the structure of the potential. Using this procedure we find the following solution of the problem:

$$\xi_1(t,x) - \tfrac{1}{2}\tau_0(t,x) = \theta(t) \tag{14}$$

$$\xi_0(t,x) - f_1(t,x) = -\dot{\theta}(t)x + \varphi(t) \tag{15}$$

$$\Pi(t,x)\tau_0(t,x) + f_0(t,x) = -\tfrac{1}{2}\ddot{\theta}(t)x^2 + \dot{\varphi}(t)x - 2\theta(t)[\Pi(t,x) + \psi(t)] \tag{16}$$

where $\theta(t)$, $\varphi(t)$ and $\psi(t)$ are arbitrary functions of the time variable obtained in the course of integration with respect to variable x. Finally, by substituting expressions (15) and (16) into (13) we obtain a partial differential equation which fully determines the potential

$$2\theta(t)\frac{\partial \Pi(t,x)}{\partial t} + [\dot{\theta}(t)x - \varphi(t)]\frac{\partial \Pi(t,x)}{\partial x} \\ +2\dot{\theta}(t)\Pi(t,x) + \tfrac{1}{2}\dddot{\theta}(t)x^2 - \ddot{\varphi}(t)x + 2\frac{\mathrm{d}}{\mathrm{d}t}[\theta(t)\psi(t)] = 0. \tag{17}$$

Having found the solution of equation (17), which will be referred to throughout the paper as the potential equation, we simultaneously fix the structure of the dynamical system and determine its conservation law. Of course, there are some other possible methods for dealing with this problem, in particular when we consider the potential equation as a kind of compatibility condition that has to be satisfied if an invariant is to be constructed. Our solution leads at least to a lowest-order potential equation, which is very important since, wherever possible, one tries to find a general solution of the problem. In the rest of the paper we shall rely on the results just obtained and attempt to reach our main goal, i.e. the generalization of the Lewis invariant.

4. Invariants of linear dynamical systems

In this section we shall deal with dynamical systems whose potentials have the form

$$\Pi(t, x) = \tfrac{1}{2}q(t)x^2$$

where $q(t)$ is an arbitrary differentiable function. The dynamical equation then reduces to a linear differential equation

$$\ddot{x} + q(t)x = 0. \tag{18}$$

We have not imposed any non-negativity restriction on the dynamical parameter $q(t)$, which demonstrated our intention to cover a larger class of systems than the oscillatory ones. In this case the potential equation (17) reduces to the form

$$\tfrac{1}{2}[\dddot{\theta}(t) + 4q(t)\dot{\theta}(t) + 2\dot{q}(t)\theta(t)]x^2 - [\ddot{\varphi}(t) + q(t)\varphi(t)]x + 2\frac{\mathrm{d}}{\mathrm{d}t}[\theta(t)\psi(t)] = 0$$

which can be decomposed into a set of three ordinary differential equations in order to be satisfied identically,

$$\dddot{\theta}(t) + 4q(t)\dot{\theta}(t) + 2\dot{q}(t)\theta(t) = 0 \tag{19}$$

$$\ddot{\varphi}(t) + q(t)\varphi(t) = 0 \tag{20}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[\theta(t)\psi(t)] = 0. \tag{21}$$

It is rather obvious that the solution of equation (21) which reads $\theta(t)\psi(t) = \text{constant}$ contributes to the conservation law (9) only a constant term which could be omitted (see equation (16)). Therefore, without loss of generality we can suppose that $\psi(t) = 0$. It is also inevitable that equation (20) possesses the same form as the dynamical equation (18). For the construction of a quadratic invariant it is crucial to find a non-trivial solution at least for $\theta(t)$, which means that we could choose $\varphi(t) = 0$ as a particular solution of equation (20). Thus, we can conclude that the dynamical system (18) possesses the quadratic conservation law

$$I = \theta(t)\dot{x}^2 - \dot{\theta}(t)x\dot{x} + \tfrac{1}{2}[\ddot{\theta}(t) + 2q(t)\theta(t)]x^2 = \text{constant} \tag{22}$$

where $\theta(t)$ represents any solution of equation (19). One can easily recognize that equation (19) plays the role of the auxiliary equation in this context. Moreover, further analysis will show that it can be reduced to a well known auxiliary equation which serves for the construction of the Lewis invariant, and can also be used for developing an Ermakov-type procedure in the present situation.

4.1. The Lewis invariant

The problem of constructing the Lewis invariant using Noether's theorem has already been tackled by Ray and Reid (1979b). It was shown that by a simple substitution, $\theta(t) = \rho^2(t)$, a straightforward calculation reduces equation (19) to a familiar form

$$\ddot{\rho}(t) + q(t)\rho(t) = \frac{k}{\rho^3(t)} \tag{23}$$

where k is a constant of integration. Thus, the dynamical system (18) has a conservation law

$$I = [\rho(t)\dot{x} - \dot{\rho}(t)x]^2 + k\left[\frac{x}{\rho(t)}\right]^2 = \text{constant} \tag{24}$$

where $\rho(t)$ represents any solution of the auxiliary equation (23). It is obvious that for $k = 1$ equation (24) reduces to the original Lewis invariant. Now, we come to our main point. Keeping in mind the procedure that was followed in this and previous sections, we can state emphatically that in the one-dimensional approach the auxiliary equation is an outcome of the condition of invariance (7). This fact will motivate our further discussions of possible generalizations of the Lewis invariant. Namely, we shall seek other possible solutions of the potential equation which will lead to the same form of auxiliary equation.

4.2. A generalization of the Ermakov-type procedure

The original Ermakov procedure for deriving a quadratic invariant was based on the assumption that equations (18) and (23) are fully equivalent. A simple elimination of the dynamical parameter $q(t)$ led to an invariant (24). Our intention is to apply the Ermakov-type procedure to the original auxiliary equation (19) and discuss certain types of linear dynamical systems and their first integrals. In order to eliminate the dynamical parameter from the problem we shall treat equation (19) as a linear differential equation of first order with respect to $q(t)$

$$\dot{q}(t) + 2\frac{\dot{\theta}(t)}{\theta(t)}q(t) = -\frac{1}{2}\frac{\ddot{\theta}(t)}{\theta(t)}$$

the solution to which is (see Ince 1956)

$$q(t) = \lambda\theta^{-2}(t) - \tfrac{1}{2}\theta^{-2}(t)\int \ddot{\theta}(t)\theta(t)\,\mathrm{d}t$$

where λ is a constant of integration. Thus, we can conclude that the dynamical system

$$\ddot{x} + \left[\lambda\theta^{-2}(t) - \tfrac{1}{2}\theta^{-2}(t)\int \ddot{\theta}(t)\theta(t)\,\mathrm{d}t\right]x = 0 \tag{25}$$

has a quadratic invariant

$$I = \theta(t)\dot{x}^2 - \dot{\theta}(t)x\dot{x} + \tfrac{1}{2}\left\{\ddot{\theta}(t) + 2\lambda\theta^{-1}(t) - \theta^{-1}(t)\int \ddot{\theta}(t)\theta(t)\,\mathrm{d}t\right\}x^2 = \text{constant}. \tag{26}$$

As far as the author is aware, this is the first time that this type of result has appeared in the literature.

The linear dynamical system (25) covers a rather broad class of systems that are important in physics. Let us give a few examples. For $\theta(t) = t$ equation (25) reduces to a special case of the Euler differential equation

$$\ddot{x} + \lambda t^{-2}x = 0$$

and from (26) we obtain its first integral

$$I = t\dot{x}^2 - x\dot{x} + \lambda t^{-1}x^2 = \text{constant}.$$

For the slightly general choice of function $\theta(t) = t^{\alpha}$, $\alpha = $ constant, we obtain a Bessel-type differential equation

$$\ddot{x} + \left[\lambda t^{-2\alpha} - \tfrac{1}{4}\alpha(\alpha - 2)t^{-2}\right]x = 0$$

which, according to (26), possesses a quadratic invariant

$$I = t^{\alpha}\dot{x}^2 - \alpha t^{\alpha-1}x\dot{x} + \left[\lambda t^{-\alpha} + \tfrac{1}{4}\alpha^2 t^{\alpha-2}\right]x^2 = \text{constant}.$$

It is interesting to note that this last differential equation is completely solvable and its solution can be expressed in terms of cylindrical functions (see Kamke 1959)

$$x(t) = t^{1/2} Z_{1/2}\left(\frac{\lambda^{1/2}}{1-\alpha} t^{1-\alpha}\right).$$

One can note that the quadratic invariant of the dynamical system has a compact form despite the fact that the general solution is expressed in terms of infinite series.

4.3. An example of decomposition

We have tried throughout this section to emphasize the role of the auxiliary equation in its original form (19), and in our concluding remarks we shall show an example of the application of the decomposition method to find its particular solution. Namely, we can transform equation (19) in the following way:

$$\frac{\mathrm{d}}{\mathrm{d}t}[\ddot{\theta}(t) + q(t)\theta(t)] + 3q(t)\dot{\theta}(t) + \dot{q}(t)\theta(t) = 0$$

and then decompose it into a set of two ordinary differential equations

$$\ddot{\theta}(t) + q(t)\theta(t) = C = \text{constant} \tag{27}$$

$$3q(t)\dot{\theta}(t) + \dot{q}(t)\theta(t) = 0. \tag{28}$$

Equation (28), which can be easily integrated, gives us a relation between $q(t)$ and $\theta(t)$

$$q(t)\theta^3(t) = D = \text{constant}. \tag{29}$$

By substituting this result into equation (27) we can solve it by quadratures and obtain

$$\frac{\sqrt{2}}{2}\int\left[\frac{\theta}{C\theta^2 + E\theta + D}\right]^{1/2} \mathrm{d}\theta = t$$

where E = constant, and the additive integration constant is omitted. The last equation defines a function $\theta(t)$ through an inverse function. Namely, the relation $t = h(\theta)$ implies $\theta(t) = h^{-1}(t)$ (h^{-1} denoting the inverse of h), if it exists. Finally, the dynamical parameter $q(t)$ is determined by equation (29), which completes our analysis. Certainly, this method of decomposition is not the only one but it is nevertheless paradigmatic since it uncovers another possible way of dealing with the problem which includes the auxiliary equation.

5. An intermediate result: a general solution of the potential equation

Let us try to find a general solution of the potential equation (17). In order to simplify it let us restrict ourselves from the outset to the case where $\varphi(t) = \psi(t) = 0$ which leads to

$$2\theta(t)\frac{\partial \Pi(t,x)}{\partial t} + \dot{\theta}(t)x\frac{\partial \Pi(t,x)}{\partial x} = -2\dot{\theta}(t)\Pi(t,x) - \tfrac{1}{2}\ddot{\theta}(t)x^2. \tag{30}$$

A general solution of a partial differential equation of the first order (30) can be constructed if we are able to find two independent first integrals of the equivalent system of the first-order ordinary differential equations (Stepanov 1945)

$$\frac{\mathrm{d}t}{2\theta(t)} = \frac{\mathrm{d}x}{\dot{\theta}(t)x} = \frac{\mathrm{d}\Pi}{-2\dot{\theta}(t)\Pi - \frac{1}{2}\ddot{\theta}(t)x^2}. \tag{31}$$

Unfortunately, under the assumption $\dot{\theta}(t) \neq 0$ we can obtain only one first integral of the system (31) which does not contain the potential $\Pi(t, x)$

$$\frac{x}{\sqrt{\theta(t)}} = C_1 = \text{constant}. \tag{32}$$

This situation can be resolved by introducing a further restriction, i.e. $\dddot{\theta}(t) = 0$, which implies

$$\theta(t) = \theta_2 t^2 + \theta_1 t + \theta_0 \qquad \theta_i = \text{constant} \quad i = 0, 1, 2. \tag{33}$$

For $\theta(t)$ defined by equation (33) we can determine two other first integrals of the system (31)

$$\theta(t)\Pi(t, x) = C_2 = \text{constant} \tag{34}$$

$$x^2\Pi(t, x) = C_3 = \text{constant}. \tag{35}$$

It must be stressed that these first integrals are not functionally independent since $C_3 = C_1^2 C_2$. This leads to the conclusion that we can construct general solutions of equation (30) under the assumption (33) in three different ways which are supposed to be equivalent. We shall now discuss each of these cases.

Case 1. By using the first integrals (32) and (34) we can construct the general solution of equation (30) in the following form:

$$\Pi(t, x) = \frac{1}{\theta(t)} F\left(\frac{x}{\sqrt{\theta(t)}}\right) \tag{36}$$

where $F(u)$ is an arbitrary differentiable function, which implies that the dynamical system

$$\ddot{x} = -\theta^{-3/2}(t) F'\left(\frac{x}{\sqrt{\theta(t)}}\right)$$

where $F'(u) = \mathrm{d}F(u)/\mathrm{d}u$, has a conservation law

$$I = \theta(t)\dot{x}^2 - \dot{\theta}(t)x\dot{x} + \tfrac{1}{2}\ddot{\theta}(t)x^2 + 2F\left(\frac{x}{\sqrt{\theta(t)}}\right) = \text{constant}.$$

This result is known in the literature (see Kamke 1959).

Case 2. Let us combine the first integrals (32) and (35) to obtain another form of the general solution

$$\Pi(t, x) = \frac{1}{x^2} \Phi\left(\frac{x}{\sqrt{\theta(t)}}\right) \tag{37}$$

where $\Phi(u)$ is an arbitrary differentiable function. In this case the dynamical system

$$\ddot{x} = \frac{2}{x^3} \Phi\left(\frac{x}{\sqrt{\theta(t)}}\right) - \frac{1}{x^2\sqrt{\theta(t)}} \Phi'\left(\frac{x}{\sqrt{\theta(t)}}\right)$$

where $\Phi'(u) = \mathrm{d}\Phi(u)/\mathrm{d}u$, possesses a quadratic invariant

$$I = \theta(t)\dot{x}^2 - \theta(t)x\dot{x} + \frac{1}{2}\ddot{\theta}(t)x^2 + 2\frac{\theta(t)}{x^2} \Phi\left(\frac{x}{\sqrt{\theta(t)}}\right) = \text{constant}.$$

It can be recognized that potentials (37) and (36) are related in the following way:

$$\Phi\left(\frac{x}{\sqrt{\theta(t)}}\right) = \frac{x^2}{\theta(t)} F\left(\frac{x}{\sqrt{\theta(t)}}\right)$$

which emphasizes the full equivalence of these two cases, without any regard to their formal differences.

Case 3. This case is perhaps the most intriguing. Since the first integrals (34) and (35) both contain the unknown function $\Pi(t, x)$, a general solution can be given only in an implicit way, i.e. it can be obtained as a solution of the following equation:

$$\Theta[\theta(t)\Pi(t, x), x^2\Pi(t, x)] = 0 \tag{38}$$

where $\Theta(\xi, \eta)$ is an arbitrary function of two independent variables. One can conclude that the dynamical system

$$\ddot{x} = -\frac{\partial \Pi(t, x)}{\partial x}$$

has a conservation law

$$I = \theta(t)\dot{x}^2 - \dot{\theta}(t)x\dot{x} + \tfrac{1}{2}\ddot{\theta}(t)x^2 + 2\theta(t)\Pi(t, x) = \text{constant}$$

where the potential $\Pi(t, x)$ is defined by equation (38). Although this result is assumed to be equivalent to previous ones, it also includes the possibility of covering a broader class of solutions to the problem than in the first two cases. This speculation stems from the fact that full equivalence can be proved if and only if the implicit function theorem is applicable. Since this theorem is of local character, there is a good reason to expect that solutions of equation (38) include solutions described in previous cases, but also cover different ones. It must be noted that an implicit way of defining the potential is known in the literature. For example, Giacomini (1990) used this method in the analysis of one-dimensional time-dependent Hamiltonian systems where the invariants possessed the form of higher transcendental functions in momentum.

6. Generalized Lewis invariants

Let us now proceed with the study of generalized Lewis invariants. As was anticipated earlier in the paper, we shall seek possible solutions to the problem of the auxiliary equation with the same form as in the original Lewis invariant. This goal can be achieved with the following form of the potential:

$$\Pi(t, x) = \tfrac{1}{2}q(t)x^2 + V(t, x).$$

With this assumption, the potential equation (30) reduces to the form

$$\tfrac{1}{2}[\dddot{\theta}(t) + 4q(t)\dot{\theta}(t) + 2\dot{q}(t)\theta(t)]x^2 + 2\theta(t)\frac{\partial V(t, x)}{\partial x} + \dot{\theta}(t)x\frac{\partial V(t, x)}{\partial t} + 2\dot{\theta}(t)V(t, x) = 0$$

which can be easily decomposed into the following system:

$$\dddot{\theta}(t) + 4q(t)\dot{\theta}(t) + 2\dot{q}(t)\theta(t) = 0 \tag{39}$$

$$2\theta(t)\frac{\partial V(t, x)}{\partial t} + \dot{\theta}(t)x\frac{\partial V(t, x)}{\partial x} = -2\dot{\theta}(t)V(t, x). \tag{40}$$

It is clear that equation (39) is identical to the auxiliary equation (19), while the partial differential equation (40) has a very tractable form which enables us to construct its general solution. Namely, the equivalent system of ordinary differential equations

$$\frac{\mathrm{d}t}{2\theta(t)} = \frac{\mathrm{d}x}{\dot{\theta}(t)x} = \frac{\mathrm{d}V}{-2\dot{\theta}(t)V}$$

possesses the first integrals

$$\frac{x}{\sqrt{\theta(t)}} = C_1 = \text{constant} \tag{41}$$

$$\theta(t)V(t,x) = C_2 = \text{constant} \tag{42}$$

$$x^2 V(t,x) = C_3 = \text{constant} \tag{43}$$

which lead to three different forms of the general solution of equation (40). Despite the obvious similarity with the first integrals (32), (34) and (35) obtained in the previous section, a substantial difference between them is apparent. In the analysis of the general solution of the potential equation (30), we assumed that the function $\theta(t)$ has the form proposed by equation (33). In the present case $\theta(t)$ represents any solution of the auxiliary equation (39), certainly a much larger class of functions than the one described by equation (33) due to the arbitrariness of the dynamical parameter $q(t)$. It is also demonstrated that a simple transformation $\theta(t) = \rho^2(t)$ reduces the auxiliary equation (39) to a Lewis-type one (23). Thus, we can now analyse three different forms of general solution which will give us three different types of generalized Lewis invariants. These invariants will be related to certain Ermakov systems.

Case 1. By combining first integrals (41) and (42) we obtain the general solution of equation (40)

$$V(t,x) = \frac{1}{\rho^2(t)} F\left(\frac{x}{\rho(t)}\right) \tag{44}$$

which leads to a conclusion that the dynamical system

$$\ddot{x} + q(t)x = -\frac{1}{\rho^3(t)} F'\left(\frac{x}{\rho(t)}\right)$$

where $F(u)$ is an arbitrary differentiable function and $F'(u) = \mathrm{d}F(u)/\mathrm{d}u$, has a quadratic invariant with the form

$$I = [\rho(t)\dot{x} - \dot{\rho}(t)x]^2 + k\left[\frac{x}{\rho(t)}\right]^2 + 2F\left(\frac{x}{\rho(t)}\right) = \text{constant}.$$

This kind of problem was tackled by Ray and Reid (1979b) in a similar manner but their analysis failed to produce a result of such generality. Kaushal and Korsch (1981) discussed the problem using Lie-group methods and arrived at an equation equivalent to (40). They obtained the solution (44) by using a trial function, not emphasizing that it represents a general solution.

Case 2. If we use the first integrals (41) and (43) we obtain the following form of general solution:

$$V(t,x) = \frac{1}{x^2}\Phi\left(\frac{x}{\rho(t)}\right).$$

Thus, the dynamical system

$$\ddot{x} + q(t)x = \frac{2}{x^3}\Phi\left(\frac{x}{\rho(t)}\right) - \frac{1}{x^2\rho(t)}\Phi'\left(\frac{x}{\rho(t)}\right)$$

where $\Phi(u)$ is an arbitrary differentiable function and $\Phi'(u) = \mathrm{d}\Phi(u)/\mathrm{d}u$, has a quadratic conservation law

$$I = [\rho(t)\dot{x} - \dot{\rho}(t)x]^2 + k\left[\frac{x}{\rho(t)}\right]^2 + 2\left[\frac{\rho(t)}{x}\right]^2 \Phi\left(\frac{x}{\rho(t)}\right) = \text{constant}.$$

Once again, it can be recognized that this case is closely related to the previous one since

$$\Phi\left(\frac{x}{\rho(t)}\right) = \left[\frac{x}{\rho(t)}\right]^2 F\left(\frac{x}{\rho(t)}\right)$$

which confirms their full equivalence.

Case 3. Finally, first integrals (42) and (43) determine a general solution of equation (40) in an implicit way, as a solution of the equation

$$\Theta[\rho^2(t)V(t,x), x^2 V(t,x)] = 0 \tag{45}$$

where $\Theta(\xi, \eta)$ is an arbitrary function of two independent variables. This lead to the conclusion that the dynamical system

$$\ddot{x} + q(t)x = -\frac{\partial V(t,x)}{\partial x}$$

possesses a quadratic invariant with the form

$$I = [\rho(t)\dot{x} - \dot{\rho}(t)x]^2 + k\left[\frac{x}{\rho(t)}\right]^2 + 2\rho^2(t)V(t,x) = \text{constant}$$

where $V(t,x)$ is a solution of equation (45). The comments given at the end of the analysis of case 3 in the previous section are equally valid in this situation. Namely, we claim that this result is of a more general character than the results obtained in the first two cases, and that it uncovers a broader class of Ermakov systems than was previously known. Let us also remember that in all these cases $\rho(t)$ represents any solution of the auxiliary equation (23).

Our final remark is concerned with another possible method for obtaining the same results. Straightforward calculation shows that we can arrive at the same set of results if the form of the potential is assumed to be as follows:

$$\Pi(t,x) = \tfrac{1}{2}q(t)x^2 + V(\theta(t), x).$$

In this analysis we obtain the same partial differential equation as was actually obtained by Kaushal and Korsch (1981).

One could complain that the analysis performed in this section has led to Ermakov systems in which the auxiliary equation has retained its original form, known from the Lewis invariant, while the dynamical equation has suffered a serious change in its structure since a nonlinear coupling with the auxiliary equation now appears. From the point of view of Ermakov systems this complaint is somewhat irrelevant since these equations are treated equivalently in this approach. On the other hand, the so-called auxiliary equation (23) has great physical significance as it arises in the study of travelling wave solutions associated with the Schrödinger equation, nonlinear elastodynamics and rotating shallow water theory (Rogers and Ramgulam 1989). Thus, exchanging the roles of the dynamical and auxiliary equations we can obtain a set of useful results concerning the nonlinear dynamical system (23).

7. Conclusions

In this paper possible generalizations of the Lewis invariant have been analysed. The study was based on application of the theorem of Noether. In our analysis a one-dimensional approach has been adopted and a Lagrangian function with a standard form of kinetic potential has been used. From this standpoint, we conclude that in the classical Lewis invariant the auxiliary equation arises as a consequence of the condition of invariance of Hamilton's action integral, and we have generalized the Ermakov-type procedure for the study of linear dynamical systems. The study of nonlinear Ermakov systems in which the auxiliary equation retains its form has also been performed. For the first time we have obtained Ermakov systems with implicitly defined potentials which are assumed to cover a broader class of dynamical systems than previously known ones.

Since this paper is concerned with the construction of certain types of quadratic invariants, Noether's theorem has been used as the primary tool for their derivation. A thorough analysis of the structure of generalized variational symmetry itself is postponed for future study. Finally, it could be of interest, in further studies, to discuss Ermakov systems within a Noetherian approach from the standpoint of two-dimensional dynamical systems.

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