PERGAMON

International Journal of Non-Linear Mechanics 35 (2000) 333–345

INTERNATIONAL JOURNAL OF
NON-LINEAR
MECHANICS

www.elsevier.com/locate/ijnonlinmec

Cubic invariants of one-dimensional Lagrangian systems☆

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Received 17 October 1998; accepted 29 March 1999

Abstract

In this paper we consider conservation laws of the third degree with respect to $\dot{x}$ of the one-dimensional time-dependent Lagrangian systems $\ddot{x} = -\partial\Pi/\partial x$. The analysis is based on the Noetherian approach. It is shown that the existence of conservation laws, as well as their structure depend on the solution of a system of first-order partial differential equations – so-called generalized Killing's equations. It is demonstrated that due to specific structure of the generators of infinitesimal transformations a rather general algorithm for derivation of cubic invariants could be established. Several types of potential $\Pi(t, x)$ which admit the existence of cubic invariants are determined. © 1999 Published by Elsevier Science Ltd. All rights reserved.

Keywords: Conservation laws; First integrals; Lagrangian systems

1. Introduction

The problem of finding conservation laws (first integrals, invariants) of dynamical systems is an old one, but its significance is so great that there have to be made a very little effort to justify the analysis directed in this way. Existence of sufficient number of conservation laws leads to a complete integrability of dynamical system (in a classical sense), which is one of the most intriguing questions in contemporary scientific research. In the classical mechanics there is a vast literature devoted to this problem. It is worth noting that in most of these analyses only linear and quadratic invariants with respect to the generalized velocities (or generalized momenta) were considered.

Besides the fact that polynomial (higher-order) invariants remained out of the scope of scientists for a long time, there have been famous examples of proving complete integrability of dynamical systems by constructing a higher-order first integral. Probably the most celebrated one is the quartic first integral of dynamically asymmetrical gyroscope, known as the top of Sophia Kowalevskaya (see Ref. [1]). Recently, construction of a complete set of first integrals, most of which are polynomial, initiated intensive study of the behavior of the systems of particles with exponential type of interaction, the so-called Toda-lattices [2]. However, it must be noted that a considerable interest in polynomial invariants arose in past few years in the fields of theoretical physics and quantum mechanics. For example, Cleary [3] analyzed the problem of non-existence and existence of higher-order first integrals for dynamical systems with polynomial potentials; Thompson [4] and Kaushal et al. [5]

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PII: S0020-7462(99)00021-9

studied the problem of constructing cubic and quartic invariants, while Grammaticos et al. [6] analyzed connection of higher-order integrals to the concept of "weak-Painlevé" property. Recently, Evans [7] proved the existence of non-separable Hamiltonians with first integrals of the fourth degree by using the method of Lax pairs, and Abraham-Shrauner [8] examined invariants of the famous Hénon–Heiles system from the group-theoretical point of view.

A brief review of the subject shows that it permanently attracts considerable attention besides possible ambiguity of the physical meaning of the polynomial invariants. Common characteristic of all above cited references is that they are primarily devoted to the analysis of two-dimensional autonomous Hamiltonian systems, and directed towards the search for the time-independent first integrals. Since the existence of the conservation law other than the Hamiltonian itself immediately leads to the complete integrability of the system, motivation for the research could be easily found. An excellent review on this subject could be found in Ref. [9]. However, there have been very few attempts to study the higher-order first integrals of one-dimensional dynamical systems. Airault [10] have performed a detailed analysis of polynomial invariants of non-autonomous dynamical systems and gave a lot of examples of the conservation laws of the fourth and the sixth degree. He emphasized that no quadratic invariants for the systems in consideration have been found. In Ref. [11] Airault established a procedure for the systematic search for polynomial invariants and showed one example of cubic invariant. The most recent article on this subject is one of Vujanovic et al. [12]. They studied the existence of quartic invariants of the generalized Emden–Fowler equation by using the well-known method based on the theorem of Emmy Noether.

The purpose of this paper is to begin the systematic study of the polynomial conservation laws of non-autonomous one-dimensional Lagrangian systems. Precisely, we shall analyze the dynamical system

$$\ddot{x} = -\frac{\partial \Pi}{\partial x}, \tag{1.1}$$

where $\Pi(t, x)$ denotes the potential of the system and an overdot denotes the derivative with respect to the independent variable t, and try to determine the form of the potential in such a way that it admits the existence of the invariant cubic with respect to $\dot{x}$. Since the dynamical system possesses the structure of an Euler–Lagrange differential equation for Lagrangian given in the form of kinetic potential

$$L(t, x, \dot{x}) = \tfrac{1}{2}\dot{x}^2 - \Pi(t, x), \tag{1.2}$$

we find it suitable to study the conservation laws through the Noetherian approach. In the following text, we shall give an outline of the classical Noether's theorem and perform the appropriate analysis. It will be shown that all cubic invariants of system (1.1) possess a common property that will come out by virtue of Noetherian analysis, and several concrete examples will be given. Differentiability of functions is assumed to be of sufficiently high order.

2. Noether's theorem: an outline

In this section we shall briefly summarize the basic ideas and results that are consisted in the theorem of Emmy Noether [13]. Namely, the essence of her discovery lays in establishing a strong relationship between conservation laws of dynamical system and invariant properties of Hamilton's action integral with respect to infinitesimal transformations of generalized coordinates and time.

It is well known that Hamilton's variational principle plays a central role in contemporary analytical mechanics and theoretical physics. Since it unifies diverse physical phenomena, its range of application goes far beyond the frontiers of classical mechanics. In present analysis we shall consider an one-dimensional dynamical system which is completely described by a Lagrangian function $L(t, x, \dot{x})$. Let us suppose that its position is specified at two instants of time: $x(t_0) = x_0$, $x(t_1) = x_1$, and let us denote the actual trajectory of the system by $x(t)$. By an actual trajectory (path) we shall assume the trajectory that joins initial and final position of the system and satisfies differential equation of motion. Along with the actual path we shall also consider a family of varied ones $\bar{x}(t)$ described by

$$\bar{x}(t) = x(t) + \delta x(t) \tag{2.1}$$

which satisfy the same boundary conditions. In Eq. (2.1) $\delta x(t)$ denotes Lagrangian (infinitesimal) variation of the function. Actually, Lagrangian variation represents the difference between the arbitrary varied path and actual one for the same value of independent variable. Finally, let us remind that the Hamilton's action integral is a functional of the following form:

$$J = \int_{t_0}^{t_1} L(t, x, \dot{x})\,\mathrm{d}t. \tag{2.2}$$

Hamilton's variational principle states that among all admissible trajectories (varied paths) the actual one makes the action integral stationary, i.e.

$$\delta J = \delta \int_{t_0}^{t_1} L(t, x, \dot{x})\,\mathrm{d}t = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} \right) \delta x = 0. \tag{2.3}$$

Due to arbitrariness of Lagrangian variation of the trajectory, as a consequence of necessary condition for extremum we obtain a differential equation of motion of the system in the form of Euler–Lagrange equation. This result immediately focuses our attention to a Lagrangian function which serves for the complete description of the behavior of the system.

In order to analyze invariant properties of Hamilton's action integral let us introduce another type of variation, termed generalized (asynchronous) variation. In this class of variations an infinitesimal variation of the independent variable is incorporated

$$\bar{t} = t + \Delta t, \tag{2.4}$$

so that we could define a generalized variation of the function in the following manner:

$$\bar{x}(\bar{t}) = x(t) + \Delta x(t). \tag{2.5}$$

After some simple manipulations and retaining the first-order infinitesimal terms we obtain the following relation between Lagrangian and generalized variation:

$$\Delta x(t) = \delta x(t) + \dot{x}(t)\Delta t. \tag{2.6}$$

According to the main goal of this section one could say that the action integral Eq. (2.2) is gauge-invariant if the following relation holds:

$$\Delta J = \Delta \int_{t_0}^{t_1} L(t, x, \dot{x})\,\mathrm{d}t = \varepsilon \int_{t_0}^{t_1} \frac{\mathrm{d}}{\mathrm{d}t} f(t, x)\,\mathrm{d}t \tag{2.7}$$

where ε is a small parameter. The action integral is said to be absolutely invariant if $f(t, x) = 0$. Let us finally suppose that generalized variations of time t and coordinate x are not arbitrary, i.e. let them be proposed as follows:

$$\Delta x = \varepsilon \xi(t, x), \quad \Delta t = \varepsilon \tau(t, x). \tag{2.8}$$

Then, Eq. (2.7) leads to the functional relation

$$L\dot{\tau} + \frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial x}\xi + \frac{\partial L}{\partial \dot{x}}(\dot{\xi} - \dot{x}\dot{\tau}) - \dot{f} = 0 \tag{2.9}$$

which expresses the condition of invariance of the action integral. This equation is usually called the basic Noether identity [14] or Noether–Bessel–Hagen equation [15]. Functions $\xi(t, x)$ and $\tau(t, x)$ are generators of the infinitesimal transformations and $f(t, x)$ is a gauge function. In the final step of this procedure we shall suppose that our system satisfies Euler–Lagrange differential equation so that after some transformations of Eq. (2.9) we arrive at the following conservation law:

$$L\tau + \frac{\partial L}{\partial \dot{x}}(\xi - \dot{x}\tau) - f = \mathrm{const.} \tag{2.10}$$

Now, we can formulate Noether's theorem: to every infinitesimal transformation of coordinate and time which leaves Hamilton's action integral absolute or gauge-invariant, there corresponds a conservation law (2.10) of the Lagrangian dynamical system. An interested reader could find a detailed proof of Noether's theorem in Ref. [14].

At this stage a few notes are ought to be given. Since Noether's theorem, in its original version, does not offer any suggestion how to find generators of infinitesimal transformations and gauge function that leaves the action integral invariant, Vujanovic [16] established a procedure for decomposition of a basic Noether identity into a set of first-order partial differential equations, termed generalized Killing's equations. Any non-trivial solution to this system immediately leads to determination of generators and gauge function. A very important generalization of this procedure was given by Djukic [17]. Namely, he supposed that velocity-dependent transformations could be included in the analysis. This assumption dramatically enlarged the possibility for finding conservation laws and it will turn out to be crucial for the study of higher-order invariants. Finally, a variety of other possible generalizations could be found in Ref. [18].

3. Generalized Killing's equations and its solution

Let us now return to the main problem of this paper. In order to study cubic invariants of the dynamical system (1.1) described completely by Lagrangian function (1.2) a Noetherian approach, sketched in the previous section, will be applied. Thorough analysis of the problem revealed that the generators of infinitesimal transformations and the gauge function are supposed to be of the following form:

$$\tau(t,x,\dot{x}) = \tau_0(t,x) + \tau_1(t,x)\dot{x}, \tag{3.1}$$

$$\xi(t,x,\dot{x}) = A(t,x)\dot{x}^2 + B(t,x)\dot{x} + C(t,x), \tag{3.2}$$

$$f(t,x,\dot{x}) = f_0(t,x) + f_1(t,x)\dot{x} + f_2(t,x)\dot{x}^2. \tag{3.3}$$

For further analysis it is of interest to write down explicitly the form of the conservation law that is to be obtained from Eq. (2.10) according to the introduced generators

$$\begin{aligned} I = {} & [A(t,x) - \tfrac{1}{2}\tau_1(t,x)]\dot{x}^3 + [B(t,x) - \tfrac{1}{2}\tau_0(t,x) \\ & - f_2(t,x)]\dot{x}^2 + [C(t,x) - \Pi(t,x)\tau_1(t,x) \\ & - f_1(t,x)]\dot{x} - [\Pi(t,x)\tau_0(t,x) + f_0(t,x)] = \text{const.} \end{aligned} \tag{3.4}$$

By substituting Eqs. (3.1)–(3.3) into a basic Noether identity (2.9), after collecting terms of the various degree with respect to $\dot{x}$ we arrive at the following relation:

$$\begin{aligned} & \left[\frac{1}{2}\frac{\partial\tau_1}{\partial x} - \frac{\partial A}{\partial x}\right]\dot{x}^4 \\ & + \left[\frac{1}{2}\frac{\partial\tau_0}{\partial x} - \frac{\partial B}{\partial x} + \frac{\partial f_2}{\partial x} + \frac{1}{2}\frac{\partial\tau_1}{\partial t} - \frac{\partial A}{\partial t}\right]\dot{x}^3 \\ & + \left[\Pi\frac{\partial\tau_1}{\partial x} - \frac{1}{2}\frac{\partial\Pi}{\partial x}\tau_1 - \frac{\partial C}{\partial x} + \frac{\partial f_1}{\partial x}\right. \\ & \left. + \frac{1}{2}\frac{\partial\tau_0}{\partial t} - \frac{\partial B}{\partial t} + \frac{\partial f_2}{\partial t} + 3A\frac{\partial\Pi}{\partial x}\right]\dot{x}^2 \\ & + \left[\Pi\frac{\partial\tau_0}{\partial x} + \frac{\partial f_1}{\partial x} + \Pi\frac{\partial\tau_1}{\partial t} + \frac{\partial\Pi}{\partial t}\tau_1\right. \\ & \left. - \frac{\partial C}{\partial t} + \frac{\partial f_1}{\partial t} + 2B\frac{\partial\Pi}{\partial x} - 2f_2\frac{\partial\Pi}{\partial x}\right]\dot{x} \\ & + \left[\Pi\frac{\partial\tau_0}{\partial t} + \frac{\partial\Pi}{\partial t}\tau_0 + \frac{\partial f_0}{\partial t} - \Pi\tau_1\frac{\partial\Pi}{\partial x}\right. \\ & \left. + C\frac{\partial\Pi}{\partial x} - f_1\frac{\partial\Pi}{\partial x}\right] = 0. \end{aligned} \tag{3.5}$$

Since the bracketed terms are independent of $\dot{x}$, Eq. (3.5) could be decomposed into a set of five first-order partial differential equations – generalized Killing's equations

$$\frac{\partial}{\partial x}\left[A - \frac{1}{2}\tau_1\right] = 0, \tag{3.6}$$

$$\frac{\partial}{\partial x}\left[B - \frac{1}{2}\tau_0 - f_2\right] + \frac{\partial}{\partial t}\left[A - \frac{1}{2}\tau_1\right] = 0, \tag{3.7}$$

$$\begin{aligned} & \frac{\partial}{\partial x}[C - \Pi\tau_1 - f_1] + \frac{\partial}{\partial t}\left[B - \frac{1}{2}\tau_0 - f_2\right] \\ & - 3\left[A - \frac{1}{2}\tau_1\right]\frac{\partial\Pi}{\partial x} = 0, \end{aligned} \tag{3.8}$$

$$\begin{aligned} & \frac{\partial}{\partial x}[\Pi\tau_0 + f_0] - \frac{\partial}{\partial t}[C - \Pi\tau_1 - f_1] \\ & + 2\left[B - \frac{1}{2}\tau_0 - f_2\right]\frac{\partial\Pi}{\partial x} = 0, \end{aligned} \tag{3.9}$$

$$\frac{\partial}{\partial t}[\Pi\tau_0 + f_0] + [C - \Pi\tau_1 - f_1]\frac{\partial\Pi}{\partial x} = 0. \tag{3.10}$$

One could easily see that generalized Killing's equations are adjusted in accordance with the form of coefficients in the expression for conservation law (3.4). It is also obvious that Eqs. (3.6)–(3.9) will serve for successive determination of the coefficients, while Eq. (3.10) will impose additional constraint which have to be satisfied if cubic invariant is to be constructed.

Integration of Eqs. (3.6)–(3.9) leads to the following solution:

$$A(t,x) - \tfrac{1}{2}\tau_1(t,x) = \theta(t), \tag{3.11}$$

$$B(t,x) - \tfrac{1}{2}\tau_0(t,x) - f_2(t,x) = -\dot{\theta}(t)x + \varphi(t), \tag{3.12}$$

$$C(t,x) - \Pi(t,x)\tau_1(t,x) - f_1(t,x) = 3\theta(t)[\Pi(t,x) + \psi(t)] + \tfrac{1}{2}\ddot{\theta}(t)x^2 - \dot{\varphi}(t), \tag{3.13}$$

$$\begin{aligned}\Pi(t,x)\tau_0(t,x) + f_0(t,x) &= 2\dot{\theta}(t)\int x\frac{\partial \Pi(t,x)}{\partial x}\,\mathrm{d}x - 2\varphi(t)[\Pi(t,x) + \kappa(t)] \\ &\quad + 3\int \frac{\partial}{\partial t}[\theta(t)\Pi(t,x)]\,\mathrm{d}x + 3\frac{\mathrm{d}}{\mathrm{d}t}[\theta(t)\psi(t)]x \\ &\quad + \frac{1}{6}\dddot{\theta}(t)x^3 - \frac{1}{2}\ddot{\varphi}(t)x^2.\end{aligned} \tag{3.14}$$

The functions $\theta(t)$, $\varphi(t)$, $\psi(t)$ and $\kappa(t)$ are arbitrary functions of the independent variable that are obtained in the course of partial integration with respect to variable x. Solutions (3.11)–(3.14) of the system of generalized Killing's equations determines the most general form of the cubic invariant for the dynamical system (1.1). As it is mentioned above, the last equation (3.10) produces the functional relation between the potential $\Pi(t,x)$ and arbitrary functions $\theta(t)$, $\varphi(t)$, $\psi(t)$ and $\kappa(t)$

$$\begin{aligned}&2\frac{\partial}{\partial t}\left[\dot{\theta}(t)\int x\frac{\partial \Pi(t,x)}{\partial x}\,\mathrm{d}x\right] \\ &\quad - 2\frac{\partial}{\partial t}[\varphi(t)\Pi(t,x)] - 2\frac{\mathrm{d}}{\mathrm{d}t}[\varphi(t)\kappa(t)] \\ &\quad + 3\frac{\partial}{\partial t}\int\frac{\partial}{\partial t}[\theta(t)\Pi(t,x)]\,\mathrm{d}x + 3\frac{\mathrm{d}^2}{\mathrm{d}t^2}[\theta(t)\psi(t)]x \\ &\quad + \frac{1}{6}\theta^{IV}(t)x^3 - \frac{1}{2}\dddot{\varphi}(t)x^2 \\ &\quad + \left[3\theta(t)\Pi(t,x) + 3\theta(t)\psi(t) + \frac{1}{2}\ddot{\theta}(t)x^2 - \dot{\varphi}(t)x\right] \\ &\quad \times\frac{\partial \Pi(t,x)}{\partial x} = 0.\end{aligned} \tag{3.15}$$

Eq. (3.15) will be referred throughout the following text as the *potential equation*. Main body of the subsequent analysis will be concerned with the study of the potential equation in order to determine potential $\Pi(t,x)$ and a non-trivial set of functions $\theta(t)$, $\varphi(t)$, $\psi(t)$ and $\kappa(t)$ such that it is satisfied identically.

4. Autogenerative character of cubic invariants

Among a variety of different approaches in the study of conservation laws of dynamical systems we have chosen to base the analysis of cubic invariants on the theorem of Emmy Noether. Our opinion is that Noetherian approach could initiate a deep and interesting study because of its intimate relation to the invariance properties of the Hamilton's action integral. It could also be noticed that the coefficients in Eq. (3.4) possess a considerable structural richness. This will enable us to easily adapt the generators of the infinitesimal transformations so that a quite general algorithm for derivation of cubic invariants could be established.

Let us suppose that the independent variable t does not suffer any transformation, i.e. $\bar{t} = t$, which implies $\tau_0(t,x) = \tau_1(t,x) = 0$. In that case one could easily conclude from Eq. (3.11) that $A(t,x) = \theta(t)$. If we suppose that, $B(t,x) = 0$, then Eq. (3.12) implies $f_2(t,x) = \dot{\theta}(t)x - \varphi(t)$. Finally, if we propose that $C(t,x) = -2\theta(t)\Pi(t,x)$, then Eq. (3.13) yields the following expression for $f_1(t,x)$

$$f_1(t,x) = -5\theta(t)\Pi(t,x) - 3\theta(t)\psi(t) - \tfrac{1}{2}\ddot{\theta}(t)x^2 + \dot{\varphi}(t)x.$$

From Eq. (3.14) $f_0(t,x)$ could be directly derived. Thus, we can conclude that in order to construct a cubic invariant for any type of potential $\Pi(t,x)$ generators of the infinitesimal transformations and gauge function could be used in the following form:

$$\tau(t, x, \dot{x}) = 0, \tag{4.1}$$

$$\xi(t, x, \dot{x}) = 2\theta(t)\left[\frac{\dot{x}^2}{2} - \Pi(t, x)\right] = 2\theta(t)L(t, x, \dot{x}), \tag{4.2}$$

$$f(t, x, \dot{x}) = [\dot{\theta}(t)x - \varphi(t)]\dot{x}^2 - \left[5\theta(t)\Pi(t, x) + 3\theta(t)\psi(t) + \frac{1}{2}\ddot{\theta}(t)x^2 - \dot{\varphi}(t)x\right]\dot{x} + 2\dot{\theta}(t)\int x\frac{\partial \Pi(t, x)}{\partial x}\,\mathrm{d}x - 2\varphi(t)[\Pi(t, x) + \kappa(t)] + 3\int\frac{\partial}{\partial t}[\theta(t)\Pi(t, x)]\,\mathrm{d}x + 3\frac{\mathrm{d}}{\mathrm{d}t}[\theta(t)\psi(t)]x + \frac{1}{6}\dddot{\theta}(t)x^3 - \frac{1}{2}\ddot{\varphi}(t)x^2. \tag{4.3}$$

Even a brief inspection of Eqs. (4.1) and (4.2) reveals a very interesting characteristic of cubic invariants: time generator τ is identically equal to zero while the structure of space generator ξ is such that it consists of the Lagrangian of the system multiplied by an arbitrary function of time. It could be said that Noetherian approach discovers two-fold purpose of the Lagrangian function of the dynamical system in consideration. On the one hand, it serves for complete description of the behavior of the system due to Euler–Lagrange structure of differential equation of motion, as it was mentioned in Section 2. On the other hand, it makes a deep influence in the process of construction of conservation laws for it determines the structure of the generators of transformations which leave the Hamilton's action integral invariant. This result motivates us to state that conservation laws of the third order for the class of systems described by the Lagrangian function (1.2) have *autogenerative character*. In this analysis gauge function is a polynomial of the second order with respect to $\dot{x}$ with coefficients which are functions of time t and coordinate x that could be easily determined in the course of analysis. It have to be noted that similar property came out from the study of the fourth-degree invariants of the generalized Emden–Fowler equation in Ref. [12].

5. Rheo-linear system

Let us now concentrate on the problem of solving the potential equation (3.15). In this section we shall analyze the existence of the non-trivial solution for the potential which is quadratic with respect to generalized coordinate x

$$\Pi(t, x) = q(t)\frac{x^2}{2}. \tag{5.1}$$

Dynamical equation (1.1) then describes the behavior of the rheo-linear system

$$\ddot{x} + q(t)x = 0. \tag{5.2}$$

Eq. (5.2) is closely connected to time-dependent oscillatory systems but it also reflects many other physical phenomena described by non-autonomous second-order linear differential equation, which is in general reducible to Eq. (5.2). It have to be noted that the existence of cubic invariants for this class of systems was anticipated in the paper of Katzin and Levine [19]. After substitution of Eq. (5.1) in Eq. (3.15) we arrive at the relation which is polynomial in x

$$\{\theta^{IV}(t) + 10q(t)\ddot{\theta}(t) + 10\dot{q}(t)\dot{\theta}(t) + 3[\ddot{q}(t) + 3q^2(t)]\theta(t)\}\frac{x^3}{6} - [\dddot{\varphi}(t) + 4q(t)\dot{\varphi}(t) + 2\dot{q}(t)\varphi(t)]\frac{x^2}{2} + 3\left\{\frac{\mathrm{d}^2}{\mathrm{d}t^2}[\theta(t)\psi(t)] + q(t)\theta(t)\psi(t)\right\}x - 2\frac{\mathrm{d}}{\mathrm{d}t}[\varphi(t)\kappa(t)] = 0. \tag{5.3}$$

Since the coefficients of the polynomial do not depend on x Eq. (5.3) could be decomposed into a following set of ordinary differential equations:

$$\theta^{IV}(t) + 10q(t)\ddot{\theta}(t) + 10\dot{q}(t)\dot{\theta}(t) + 3[\ddot{q}(t) + 3q^2(t)]\theta(t) = 0, \tag{5.4}$$

$$\dddot{\varphi}(t) + 4q(t)\dot{\varphi}(t) + 2\dot{q}(t)\varphi(t) = 0, \tag{5.5}$$

$$\frac{d^2}{dt^2}[\theta(t)\psi(t)] + q(t)\theta(t)\psi(t) = 0, \tag{5.6}$$

$$\frac{d}{dt}[\varphi(t)\kappa(t)] = 0. \tag{5.7}$$

If we are in a position to find any solution of system (5.4)–(5.7) then we could construct a cubic invariant of the dynamical system (5.2). Since this problem is rather difficult we shall turn ourselves to a simpler one. Namely, our main goal (cubic invariant) will be accomplished if a non-trivial solution is find at least for $\theta(t)$. Therefore, let us suppose that $\varphi(t) = \psi(t) = \kappa(t) = 0$. Having in mind Eq. (3.4) and solution of the generalized Killing's equations we can conclude that dynamical system (5.2) possesses cubic conservation law

$$I = \theta(t)\dot{x}^3 - \dot{\theta}(t)x\dot{x}^2 + \tfrac{1}{2}[\ddot{\theta}(t) + 3q(t)\theta(t)]x^2\dot{x} - \tfrac{1}{6}[\dddot{\theta}(t) + 7q(t)\dot{\theta}(t) + 3\dot{q}(t)\theta(t)]x^3 = \text{const.}, \tag{5.8}$$

where $\theta(t)$ is any solution of Eq. (5.4), which will be called auxiliary equation.

At this stage it is of interest to discuss the significance of auxiliary equation (5.4). It is obvious that possibility of finding the invariant depends on the existence of its solution. Exact solution of the auxiliary equation leads to the exact invariant of the system, while any type of approximate solution enables us to construct the adiabatic invariant. Other possible applications come out as a reminiscence of the principle of non-linear superposition. Namely, Reid and Ray [20] analyzed application of this principle to time-dependent harmonic oscillator, discussed previously by Lewis [21], and obtained non-linear superposition law which relates general solution of the problem to any particular solution of auxiliary equation. Our opinion is that it is worth trying to analyze the system (5.2), (5.8) in the light of non-linear superposition principle. For a detailed account on this subject one could see Ref. [22].

Since auxiliary equation is the fourth-order non-autonomous equation, it is a very hard task to find its solution in a general case, i.e. for arbitrary $q(t)$. It indicates that we have to confine ourselves with any kind of its particular solution that could be determined. Thus, in the final part of this section we shall analyze three particular cases that come out from Eq. (5.4).

Case 1: Let us transform the auxiliary equation (5.4) in the following manner:

$$\frac{d^2}{dt^2}[\ddot{\theta}(t) + q(t)\theta(t)] + 9q(t)[\ddot{\theta}(t) + q(t)\theta(t)] + 2[4\dot{q}(t)\dot{\theta}(t) + \ddot{q}(t)\theta(t)] = 0. \tag{5.9}$$

This equation could be decomposed into a set of two ordinary differential equations of the second order

$$\ddot{\theta}(t) + q(t)\theta(t) = 0, \tag{5.10}$$

$$4\dot{q}(t)\dot{\theta}(t) + \ddot{q}(t)\theta(t) = 0. \tag{5.11}$$

Eq. (5.11) could be easily integrated so that we obtain

$$\theta^4(t) = \text{const.}/\dot{q}(t). \tag{5.12}$$

By substituting Eq. (5.12) into Eq. (5.10) we arrive at the following differential equation for $q(t)$:

$$\dot{q}(t)\dddot{q}(t) - \tfrac{5}{4}\ddot{q}^2(t) - 4q(t)\dot{q}^2(t) = 0. \tag{5.13}$$

Let us try to find the solution of Eq. (5.13) in the form $q(t) = \lambda t^m$ where λ and m are constants which are to be determined. In that case Eq. (5.13) reduces to

$$\lambda^2 m^2(m-1)(m-2)t^{2m-4} - \tfrac{5}{4}\lambda^2 m^2(m-1)^2 t^{2m-4} - 4\lambda^3 m^2 t^{3m-2} = 0. \tag{5.14}$$

In order to force the obtained equation to be homogeneous with respect to t we shall equalize the powers of t which appear in Eq. (5.14), i.e. $2m-4 = 3m-2$, which leads to $m = -2$. This reduces Eq. (5.14) to an algebraic equation in λ which have only one real solution different from zero: $\lambda = \frac{3}{16}$. Therefore, we can conclude that auxiliary equation (5.4) has a particular solution of the form

$$q(t) = \tfrac{3}{16}t^{-2}, \quad \theta(t) = Kt^{3/4}, \quad K = \text{const.} \tag{5.15}$$

Thus, dynamical system $\ddot{x} = -(\frac{3}{16})t^{-2}x$ has a cubic conservation law

$$I = t^{3/4}\dot{x}^3 - \tfrac{3}{4}t^{-1/4}x\dot{x}^2 + \tfrac{3}{16}t^{-5/4}x^2\dot{x} - \tfrac{1}{64}t^{-9/4}x^3 = \text{const.} \tag{5.16}$$

This result was obtained by Katzin and Levine in Ref. [19].

Case 2: Auxiliary Eq. (5.4) could also be recast in this way

$$\frac{d^2}{dt^2}[\ddot{\theta}(t) + 3q(t)\theta(t)] + 3q(t)[\ddot{\theta}(t) + 3q(t)\theta(t)] + 4[\dot{q}(t)\dot{\theta}(t) + q(t)\ddot{\theta}(t)] = 0 \tag{5.17}$$

which give us an opportunity to treat it as two independent differential equations

$$\ddot{\theta}(t) + 3q(t)\theta(t) = 0, \tag{5.18}$$

$$\dot{q}(t)\dot{\theta}(t) + q(t)\ddot{\theta}(t) = 0. \tag{5.19}$$

By integrating Eq. (5.19) we obtain

$$q(t) = K/\dot{\theta}(t), \quad K = \text{const.} \tag{5.20}$$

Substitution of Eq. (5.20) into Eq. (5.18) leads to the following differential equation for $\theta(t)$:

$$\ddot{\theta}(t) + 3K\frac{\theta(t)}{\dot{\theta}(t)} = 0. \tag{5.21}$$

This equation could be easily integrated and it turns out that power-law solution $\theta(t) = \lambda t^3$ satisfies it identically for $K = -6\lambda$. Since Eq. (5.20) gives $q(t) = -2t^{-2}$, we could say that dynamical system $\ddot{x} = 2t^{-2}x$ have a conservation law of the form

$$I = t^3\dot{x}^3 - 3t^2x\dot{x}^2 + 4x^3 = \text{const.} \tag{5.22}$$

Case 3: Let us finally suppose that $\theta(t) = K = \text{const}$. This assumption will immediately reduce the auxiliary equation to a single ordinary differential equation of the second order for $q(t)$

$$\ddot{q}(t) + 3q^2(t) = 0. \tag{5.23}$$

It is easy to obtain a solution to this problem in the form $q(t) = \lambda t^m$, where λ and m are constants. By direct substitution and forcing homogeneity with respect to t we find that $m = -2$ and $\lambda = -2$. Thus, we have obtained that dynamical system $\ddot{x} = 2t^{-2}x$ possesses a cubic invariant

$$I = \dot{x}^3 - 3t^{-2}x^2\dot{x} - 2t^{-3}x^3 = \text{const.} \tag{5.24}$$

It is interesting to emphasize that our analysis in cases 2 and 3 led to two independent cubic invariants of the same dynamical system. Therefore, we can conclude that the dynamical system $\ddot{x} = 2t^{-2}x$ is completely integrable due to existence of two independent conservation laws (5.22) and (5.24) which are of the third degree with respect to $\dot{x}$. It seems to be rather unusual that complete integrability comes out as a consequence of the existence of the higher-order invariants only.

It have to be pointed out that dynamical equations obtained in the analysis of cubic invariants of linear non-autonomous systems belong to the same class. They can be regarded as special cases of the well-known Euler's differential equation. Thorough study of general solution of this equation could be found in Ref. [23].

6. Non-linear system: power-law potential

The analysis of the cubic invariants will be continued by investigation of its existence for the power-law potential

$$\Pi(t, x) = q(t)\frac{x^{n+1}}{n+1}, \tag{6.1}$$

where $n \neq -1$ is real constant, which yields the following dynamical equation:

$$\ddot{x} = -q(t)x^n. \tag{6.2}$$

This type of potential reduces the potential equation (3.15) to the form

$$3\theta(t)q^2(t)\frac{x^{2n+1}}{n+1} + \left\{\frac{3}{(n+1)(n+2)}\frac{d^2}{dt^2}[\theta(t)q(t)] + \frac{2}{n+2}\frac{d}{dt}[\dot{\theta}(t)q(t)] + \frac{1}{2}\ddot{\theta}(t)q(t)\right\}x^{n+2} - \left\{\frac{2}{n+1}\frac{d}{dt}[\varphi(t)q(t)] + \dot{\varphi}(t)q(t)\right\}x^{n+1}$$

$$+ 3\theta(t)\psi(t)q(t)x^n$$
$$+ \frac{1}{6}\theta^{IV}(t)x^3 - \frac{1}{2}\dddot{\varphi}(t)x^2 + 3\frac{d^2}{dt^2}[\theta(t)\psi(t)]x$$
$$- 2\frac{d}{dt}[\varphi(t)\kappa(t)] = 0. \quad (6.3)$$

Different values of the constant n lead to a different systems of ordinary differential equations that have to be solved in order to construct a conservation law. For example, $n = 1$ recasts Eq. (6.3) in the form given by Eq. (5.3).

Thorough analysis of this problem have shown that a quite satisfactory study of cubic invariants could be given for $n = -\frac{1}{2}$. By equating to zero coefficients of the different powers of x we arrive at the following set of equations:

$$4\frac{d^2}{dt^2}[\theta(t)q(t)] + \frac{4}{3}\frac{d}{dt}[\dot{\theta}(t)q(t)] + \frac{1}{2}\ddot{\theta}(t)q(t) = 0, \quad (6.4)$$

$$4\frac{d}{dt}[\varphi(t)q(t)] + \dot{\varphi}(t)q(t) = 0, \quad (6.5)$$

$$3\theta(t)q^2(t) - \frac{d}{dt}[\varphi(t)\kappa(t)] = 0, \quad (6.6)$$

$$\theta(t)\psi(t)q(t) = 0, \quad (6.7a)$$

$$\theta^{IV}(t) = 0, \quad (6.7b)$$

$$\dddot{\varphi}(t) = 0, \quad (6.7c)$$

$$\frac{d^2}{dt^2}[\theta(t)\psi(t)] = 0. \quad (6.7d)$$

It is easy to recognize that Eq. (6.5) could be integrated, and thus implies the relation between $q(t)$ and $\varphi(t)$

$$q(t) = \lambda\varphi^{-5/4}(t) \quad (6.8)$$

where λ is a constant of integration. It is also obvious that Eqs. (6.7a) and (6.7d) will be satisfied identically for $\psi(t) = 0$. Finally, $\kappa(t)$ could be determined from Eq. (6.6) in terms of $q(t)$, $\theta(t)$ and $\varphi(t)$

$$\kappa(t) = \frac{3}{\varphi(t)}\int\theta(t)q^2(t)\,dt, \quad (6.9)$$

where the constant of integration have been omitted. Our problem is now reduced to a set of three differential equations (6.4), (6.7b) and (6.7c) together with additional constraint (6.8) obtained as a consequence of Eq. (6.5). It is noticeable at first glance that Eqs. (6.7b) and (6.7c) impose severe restrictions on the type of solution that could be expected. Namely, $\theta(t)$ and $\varphi(t)$ have to be functions of the following form:

$$\theta(t) = \theta_0 t^a, \quad \theta_0 = \text{const.}, \quad a \in \{0, 1, 2, 3\},$$
$$\varphi(t) = \varphi_0 t^b, \quad \varphi_0 = \text{const.}, \quad b \in \{0, 1, 2\}. \quad (6.10)$$

After substituting Eqs. (6.8) and (6.10) into (6.4) we obtain

$$\tfrac{5}{12}\lambda\theta_0\varphi_0^{-5/4}[14a^2 - 14a - 28ab + 15b^2 + 12b] \times t^{a-(5/4)b-2} = 0. \quad (6.11)$$

Since λ, θ_0, and φ_0 are arbitrary constants and we tend to satisfy Eq. (6.11) for arbitrary value of t, the expression in bracket have to be equal to zero. However, it have to be pointed out that we are seeking for the solutions a and b that belong to the sets proposed in Eq. (6.10). One can easily find four distinct solutions of Eq. (6.11)

$$a = 0, \quad b = 0, \quad (6.12a)$$
$$a = 1, \quad b = 0, \quad (6.12b)$$
$$a = 2, \quad b = 2, \quad (6.12c)$$
$$a = 3, \quad b = 2. \quad (6.12d)$$

This section will be concluded with a review of cubic invariants that are obtained by virtue of previously performed analysis.

Case 1: Eq. (6.12a) leads to the following solution of the system (6.4)–(6.7):

$$\theta(t) = \theta_0, \quad \varphi(t) = \varphi_0, \quad q(t) = \lambda\varphi_0^{-5/4},$$
$$\kappa(t) = 3\lambda^2\theta_0\varphi_0^{-7/2}t. \quad (6.13)$$

Thus, according to Eqs. (3.4) and (3.11)–(3.14) it could be concluded that dynamical system $\ddot{x} = -\lambda\varphi_0^{-5/4}x^{-1/2}$ possesses invariant of the form

$$I = \theta_0\dot{x}^3 + \varphi_0\dot{x}^2 + 6\lambda\theta_0\varphi_0^{-5/4}x^{1/2}\dot{x} + 4\lambda\varphi_0^{-1/4}x^{1/2} + 6\lambda^2\theta_0\varphi_0^{-5/2}t = \text{const.} \quad (6.14)$$

It is interesting to note that arbitrariness of constants θ_0 and φ_0 permits decomposition of

Eq. (6.14) into two independent invariants in the following manner $I=\theta_0 I_1+2\varphi_0 I_2$, where

$$I_1=\dot{x}^3+6\lambda\varphi_0^{-5/4}x^{1/2}\dot{x}+6\lambda^2\varphi_0^{-5/2}t=\text{const.},$$

$$I_2=\frac{\dot{x}^2}{2}+2\lambda\varphi_0^{-5/4}x^{1/2}=\text{const.} \tag{6.15}$$

Evidently, conservation law I_2 is quadratic and plays the role of energy integral since the system is autonomous. On the other hand, invariant I_1 is pure cubic invariant of the system in consideration. This conservation law was found earlier by Parsons Ref. [24] (see also Ref. [14] p. 103) in the analysis of a gas discharge problem.

Case 2: For (6.12b) one could obtain

$$\theta(t)=\theta_0 t,\quad \varphi(t)=\varphi_0,\quad q(t)=\lambda\varphi_0^{-5/4},$$
$$\kappa(t)=\tfrac{3}{2}\lambda^2\theta_0\varphi_0^{-7/2}t^2. \tag{6.16}$$

Therefore, dynamical system $\ddot{x}=-\lambda\varphi_0^{-5/4}x^{-1/2}$ have an invariant of the form

$$I=\theta_0 t\dot{x}^3-\theta_0 x\dot{x}^2+\varphi_0\dot{x}^2+6\lambda\theta_0\varphi_0^{-5/4}tx^{1/2}\dot{x}$$
$$-\tfrac{16}{3}\lambda\theta_0\varphi_0^{-5/4}x^{3/2}+4\lambda\varphi_0^{-1/4}x^{1/2}$$
$$+3\lambda^2\theta_0\varphi_0^{-5/2}t=\text{const.} \tag{6.17}$$

By virtue of the relation $I=\theta_0 I_1+2\varphi_0 I_2$ invariant (6.17) generates two independent conservation laws

$$I_1=t\dot{x}^3-x\dot{x}^2+6\lambda\varphi_0^{-5/4}tx^{1/2}\dot{x}-\tfrac{16}{3}\lambda\varphi_0^{-5/4}x^{3/2}$$
$$+3\lambda^2\varphi_0^{-5/2}t^2=\text{const.} \tag{6.18}$$

while I_2 is the same quadratic first integral as in Eq. (6.15). It is worth noting that these two cases produced conservation laws for the same dynamical system. However, cubic invariants obtained in the course of analysis are completely independent.

Case 3: Let us consider the solution that arise from Eq. (6.12c)

$$\theta_0(t)=\theta_0 t^2,\quad \varphi(t)=\varphi_0 t^2,$$
$$q(t)=\lambda\varphi_0^{-5/4}t^{-5/2},$$
$$\kappa(t)=-\tfrac{3}{2}\lambda^2\theta_0\varphi_0^{-7/2}t^{-4}. \tag{6.19}$$

According to the general form of conservation law (3.4) and solution of the generalized Killing's equations one could obtain that dynamical system $\ddot{x}=-\lambda\varphi_0^{-5/4}t^{-5/2}x^{-1/2}$ have an invariant

$$I=\theta_0 t^2\dot{x}^3-(2\theta_0 tx-\varphi_0 t^2)\dot{x}^2+(6\lambda\theta_0\varphi_0^{-5/4}t^{-1/2}x^{1/2}$$
$$+\theta_0 x^2-2\varphi_0 tx)\dot{x}-\tfrac{2}{3}\lambda\theta_0\varphi_0^{-5/4}t^{-3/2}x^{3/2}$$
$$+4\lambda\varphi_0^{-1/4}t^{-1/2}x^{1/2}-3\lambda^3\theta_0\varphi_0^{-5/2}t^{-2}+\varphi_0 x^2$$
$$=\text{const.} \tag{6.20}$$

In a similar manner as in previous two cases Eq. (6.20) could be rearranged in the form $I=\theta_0 I_1+2\varphi_0 I_2$ so that it implies existence of two first integrals

$$I_1=t^2\dot{x}^3-2tx\dot{x}^2+(6\lambda\varphi_0^{-5/4}t^{-1/2}x^{1/2}+x^2)\dot{x}$$
$$-\tfrac{2}{3}\lambda\varphi_0^{-5/4}t^{-3/2}x^{3/2}-3\lambda^2\varphi_0^{-5/2}t^{-2}=\text{const.},$$
$$I_2=t^2\frac{\dot{x}^2}{2}-tx\dot{x}+2\lambda\varphi_0^{-5/4}t^{-1/2}x^{1/2}+\frac{x^2}{2}=\text{const.} \tag{6.21}$$

These two invariants, where I_1 is cubic and I_2 quadratic, lead to a complete integrability of a non-autonomous dynamical system considered in this case.

Case 4: Finally, Eq. (6.12d) generates the following solution of the system (6.4)–(6.7)

$$\theta_0(t)=\theta_0 t^3,\quad \varphi(t)=\varphi_0 t^2,$$
$$q(t)=\lambda\varphi_0^{-5/4}t^{-5/2},$$
$$\kappa(t)=-\tfrac{3}{2}\lambda^2\theta_0\varphi_0^{-7/2}t^{-3}. \tag{6.22}$$

Thus, dynamical system $\ddot{x}=-\lambda\varphi_0^{-5/4}t^{-5/2}x^{-1/2}$ possesses the invariant

$$I=\theta_0 t^3\dot{x}^3-(3\theta_0 t^2x-\varphi_0 t^2)\dot{x}^2+(6\lambda\theta_0\varphi_0^{-5/4}t^{1/2}x^{1/2}$$
$$+3\theta_0 tx^2-2\varphi_0 tx)\dot{x}-6\lambda\theta_0\varphi_0^{-5/4}t^{-1/2}x^{3/2}$$
$$+4\lambda\varphi_0^{-1/4}t^{-1/2}x^{1/2}-6\lambda^2\theta_0\varphi_0^{-5/2}t^{-1}-\theta_0 x^3$$
$$+\varphi_0 x^2=\text{const.} \tag{6.23}$$

Eq. (6.23) could be transformed by using the recipe $I=\theta_0 I_1+2\varphi_0 I_2$ which yields two conservation laws

$$I_1=t^3\dot{x}^3-3t^3x\dot{x}^2+(6\lambda\varphi_0^{-5/4}t^{1/2}x^{1/2}+3tx^2)\dot{x}$$
$$-6\lambda\varphi_0^{-5/4}t^{-1/2}x^{3/2}-6\lambda^2\varphi_0^{-5/2}t^{-1}-x^3$$
$$=\text{const.} \tag{6.24}$$

while I_2 have the same form as in Eq. (6.21). Once again we obtained two independent cubic invariants for the same dynamical system.

It is interesting to notice that dynamical system discussed in last two cases could be related to the well-known Emden–Fowler equation $\zeta u'' + 2u' + a\zeta^{\nu}u^{n} = 0$, where prime denotes differentiation with respect to the independent variable ζ. By a simple change of variables (see Ref. [23]) it could be reduced to $\ddot{x} = -\lambda\varphi_0^{-5/4}t^{-5/2}x^{-1/2}$ for the following values of parameters: $n = \nu = -\frac{1}{2}, a = \lambda\varphi_0^{-5/4}$.

It should be noted that we may seek for the solutions $\theta(t)$ and $\varphi(t)$ of Eqs. (6.7b) and (6.7c) in the form of polynomials of the third and the second degree, respectively. This assumption will not bring any principal novelty in the results of analysis except for a time translation of the solution that will be obtained.

7. Non-linear system: implicitly defined potential

The concluding section will be devoted to a somewhat unusual form of the potential that admits an existence of the cubic invariant of dynamical system. Let us suppose that the potential is of the form

$$\Pi(t, x) = q(t)p(x), \tag{7.1}$$

which yields the following differential equation of motion:

$$\ddot{x} = -q(t)\frac{\mathrm{d}p(x)}{\mathrm{d}x}. \tag{7.2}$$

In order to simplify the potential equation (3.15) let us suppose that the functions $\theta(t)$, $\varphi(t)$, $\psi(t)$ and $\kappa(t)$ are of the form

$$\theta(t) = \theta_0 = \text{const.}, \quad \varphi(t) = \psi(t) = \kappa(t) = 0. \tag{7.3}$$

With Eq. (7.3) potential equation will be reduced to a rather tractable form

$$3\theta_0\left[\ddot{q}(t)\int p(x)\,\mathrm{d}x + q^2(t)p(x)\frac{\mathrm{d}p(x)}{\mathrm{d}x}\right] = 0. \tag{7.4}$$

If we suppose $\theta_0 \neq 0$, Eq. (7.4) could be treated by the method of separation of variables, i.e.

$$\frac{\ddot{q}(t)}{q^2(t)} = -\frac{p(x)[\mathrm{d}p(x)/\mathrm{d}x]}{\int p(x)\,\mathrm{d}x} = k = \text{const.} \tag{7.5}$$

Thus, we arrive at the set of two equations that determine the structure of the one-parameter family of potentials which admit the existence of cubic invariant

$$\ddot{q}(t) = kq^2(t), \tag{7.6}$$

$$p(x)\frac{\mathrm{d}p(x)}{\mathrm{d}x} = -k\int p(x)\,\mathrm{d}x. \tag{7.7}$$

General solution of Eq. (7.6) for special values of k could be expressed in terms of Weierstrass' elliptic functions (see Refs. [23,25] p. 640). Since there are neither initial nor boundary conditions specified for $q(t)$, we shall be confined with particular solution that contains no constants of integration. It is easy to check that function

$$q(t) = \frac{6}{k}t^{-2} \tag{7.8}$$

satisfies Eq. (7.6) identically.

Eq. (7.7) is much more intriguing than Eq. (7.6) since it determines the coordinate dependence of the potential, the question that motivated the whole paper. Its integro-differential structure could be avoided by a simple differentiation with respect to x which leads to

$$p(x)\frac{\mathrm{d}^2p(x)}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}p(x)}{\mathrm{d}x}\right)^2 = -kp(x). \tag{7.9}$$

This second-order ordinary differential equation could be recast into a homogeneous equation of the first order with p as an independent variable by the following change of variables $(\mathrm{d}p/\mathrm{d}x)^2 = F(p)$. Thus, Eq. (7.9) yields

$$\frac{\mathrm{d}F(p)}{\mathrm{d}p} = -2\left(k + \frac{F(p)}{p}\right) \tag{7.10}$$

which could be integrated using the standard procedure (see Ref. [26]). General solution of Eq. (7.10) is of the form

$$F(p) = \tfrac{1}{3}[bp^{-2} - 2kp], \quad b = \text{const.}, \tag{7.11}$$

where $b \neq 0$ is a constant of integration. Since $\mathrm{d}p(x)/\mathrm{d}x = \sqrt{F(p)}$, one could easily find by separation of variables that

$$x = \sqrt{3}\int \frac{p\,\mathrm{d}p}{\sqrt{b - 2kp^3}} = g(p), \tag{7.12}$$

where we have omitted an additive constant of integration. As it could be seen, Eq. (7.12) defines the potential in an implicit way. Its explicit form comes out as a consequence of invertibility of the function $g(p)$, i.e. $p(x) = g^{-1}(x)$. Integral term in Eq. (7.12) could be without difficulties related to elliptic integrals (see Ref. [25] p. 589). However, lack of any type of additional conditions led us to decision to avoid any kind of deeper study of this relation. Thus, we can conclude that dynamical system

$$\ddot{x} = -\frac{6}{k}t^{-2}\frac{\mathrm{d}p(x)}{\mathrm{d}x}, \tag{7.13}$$

where $p(x)$ is defined by Eq. (7.12), possesses the cubic invariant

$$I = \frac{1}{3}\dot{x}^3 + \frac{6}{k}t^{-2}p(x)\dot{x} + \frac{12}{k}t^{-3}\int p(x)\,\mathrm{d}x = \text{const.} \tag{7.14}$$

Similar kind of determination of the potential have already been seen in the paper of Giacomini [27] where the invariants have a form of higher transcendental functions in generalized momentum. It may also be noticed that Eq. (7.7) admits a particular solution which is linear with respect to x. However, the potential that will be obtained only recovers the cubic invariant Eq. (5.24) in case 3 of Section 5.

8. Conclusions

In this paper we have considered the existence of conservation laws of the third degree with respect to $\dot{x}$ of the Lagrangian system (1.1). Because of Euler–Lagrange structure of dynamical equations we find it appropriate to apply Noether's theorem, based upon the invariance properties of Hamilton's action integral with respect to infinitesimal transformations of space and time variables. It was shown that potential of the system $\Pi(t,x)$ have to satisfy the potential equation (3.15) in order to admit the existence of the cubic invariant. It was also demonstrated that a quite general procedure for derivation of cubic invariants could be established due to specific structure of the generators of space and time transformations. Namely, space generator contain a Lagrangian function of the system multiplied by some function of the independent variable t. This result motivated us to state that cubic invariants possess autogenerative character since infinitesimal transformations, which permits their existence, were generated by the very same Lagrangian function which serves for the description of dynamical behavior of the system through the Euler–Lagrange equation. This was achieved through a derivation and solution of the system of generalized Killing's equations. An analysis of rheo-linear and non-linear non-autonomous systems was performed and several examples of cubic conservation laws were given.

Acknowledgements

The author wishes to express his gratitude to Professor Bozidar D. Vujanovic and Professor Djordje S. Djukic for numerous stimulating discussions and encouragement during the development of this study.

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