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On the optimal shape of a Pflüger column

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Abstract – The optimal shape of a Pflüger column is determined by using Pontryagin's maximum principle. It is shown that the boundary value problem relevant for determining the optimal distribution of material (i.e. cross-sectional area function) along the column axis has simple eigenvalue. Necessary conditions for local extremum of column volume are reduced to a boundary-value problem for a single second order nonlinear differential equation. We examined singular points of this equation and formulated extremal complementary variational principles for it. The optimal cross-sectional area function is obtained by numerical integration and by Ritz method. The error of the analytical approximate solution obtained by Ritz method is also estimated. © 1999 Èditions scientifiques et médicales Elsevier SAS

optimal shape / Pontryagin's principle

1. Introduction

The problem of determining the shape of the column of greatest efficiency is indeed an old one. For simply supported column loaded by concentrated forces at its ends it was formulated by Lagrange (see (Cox, 1992), for example). The solution of the problem was obtained by many authors. We mention the work of Keller (1960) that became classic. Later Tadjbakhsh and Keller (1962) treated other boundary conditions by the method developed in Keller (1960). It was, for the first time, pointed out by Olhoff and Rasmussen (1977) that the change of boundary conditions could lead to multiple eigenvalues of the equation relevant for determining the optimal cross-sectional area function. In this case one is faced with a so called multiple eigenvalue optimization problem (Seyranian et al., 1994; Seiranyan, 1995) and the optimization problem becomes more involved.

In this work we propose to study the problem of determining the Pflüger column of greatest efficiency. A Pflüger column is a simply supported column loaded by uniformly distributed follower type of load (see Pflüger, 1975). The uniformly distributed follower load is a non-conservative load. It is interesting, however, that despite the non-conservative character of the load the stability analysis for Pflüger column could be based on static (Euler) method. In the first part of this paper we shall formulate the nonlinear system of equations describing the equilibrium configuration of a column. On the basis of these equations we shall derive the linear boundary value problem that determines the stability boundary. By analyzing the multiplicity of the lowest eigenvalue of this equation we shall confirm that it represents the bifurcation point of the nonlinear system. Then we shall formulate the optimization problem. The necessary conditions for the minimum of volume will be reduced to a single nonlinear differential equation. The rest of the paper is devoted to analysis of this equation, both numerical and variational. As results of this analysis we shall determine the optimal distribution of the material along the column axis and show that the optimal column has 19% smaller volume then the corresponding column with constant cross section.

2. Formulation of the problem

Consider a column shown in *figure 1*. The column is simply supported at both ends with end C movable. The axis of the column is initially straight and the column is loaded by uniformly distributed follower type of load of constant intensity q_0. We shall assume that the column axis has length L and that it is inextensible.

Let x-B-y be a Cartesian coordinate system with the origin at the point B and with the x axis oriented along the column axis in the undeformed state. The equilibrium equations could now be easily derived (see Atanackovic, 1997)

$$\frac{\mathrm{d}H}{\mathrm{d}S} = -q_x; \qquad \frac{\mathrm{d}V}{\mathrm{d}S} = -q_y; \qquad \frac{\mathrm{d}M}{\mathrm{d}S} = -V\cos\theta + H\sin\theta, \tag{1}$$

where H and V are components of the resultant force (a force representing the influence of the part $(S, L]$ on the part $[0, S]$ of the column) along the x and y axis, respectively, M is the bending moment and θ is the angle between the tangent to the column axis and x axis. Also in (1) q_x and q_y are components of the distributed forces along the x and y axis respectively. Since the distributed force is tangent to the column axis we have

$$q_x = -q_0\cos\theta; \qquad q_y = -q_0\sin\theta. \tag{2}$$

To the system (1) we adjoin the following geometrical

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}S} &= \cos\theta; \\ \frac{\mathrm{d}y}{\mathrm{d}S} &= \sin\theta, \end{aligned} \tag{3}$$

and constitutive relation

$$\frac{\mathrm{d}\theta}{\mathrm{d}S} = \frac{M}{EI}. \tag{4}$$

In (3) and (4) we use x and y to denote coordinates of an arbitrary point of the column axis and EI to denote the bending rigidity. The boundary conditions corresponding to the column shown in *figure 1* are

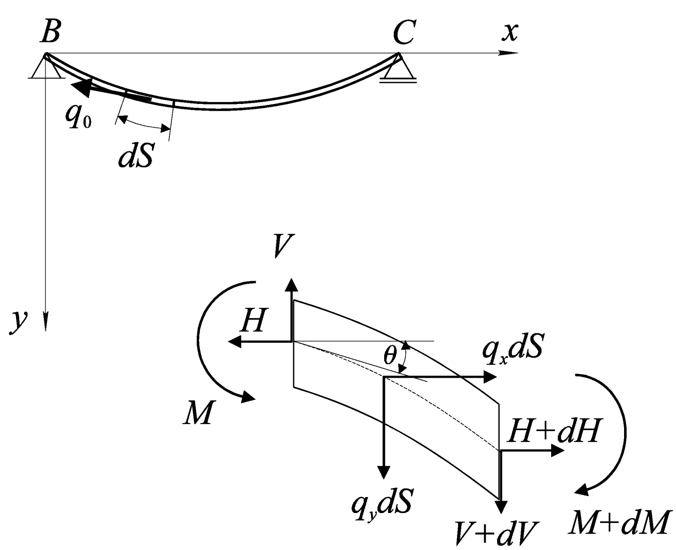

Figure 1. Coordinate system and load configuration.

$$x(0) = 0; \qquad y(0) = 0; \qquad M(0) = 0;$$
$$y(L) = 0; \qquad M(L) = 0; \qquad H(L) = 0. \tag{5}$$

The system (1)–(5) possesses a trivial solution in which column axis remains straight, i.e.,

$$H^0(S) = -q_0(L - S); \qquad V^0(S) = 0; \qquad M^0(S) = 0;$$
$$x^0(S) = S; \qquad y^0(S) = 0; \qquad \theta^0(S) = 0. \tag{6}$$

In order to formulate the minimum volume problem for the column we take the cross-sectional area $A(S)$ and the second moment of inertia $I(S)$ of the cross-section in the form

$$A(S) = A_0 a(S); \qquad I(S) = I_0 a^2(S), \tag{7}$$

where A_0 and I_0 are constants (having dimensions of area and second moment of inertia, respectively) and $a(S)$ is cross-sectional area function. For the case of a column with circular cross section we have the connection between A_0 and I_0 given by $I_0 = (1/4\pi)A_0^2$. Let $\Delta H, \ldots, \Delta\theta$ be the perturbations of $H, \ldots, \theta$ defined by

$$H = H^0 + \Delta H; \qquad V = V^0 + \Delta V; \qquad M = M^0 + \Delta M;$$
$$x = x^0 + \Delta x; \qquad y = y^0 + \Delta y; \qquad \theta = \theta^0 + \Delta\theta. \tag{8}$$

Then, by introducing the following dimensionless quantities

$$h = \frac{\Delta H L^2}{E I_0}; \qquad v = \frac{\Delta V L^2}{E I_0}; \qquad m = \frac{\Delta M L}{E I_0};$$
$$\xi = \frac{\Delta x}{L}; \qquad \eta = \frac{\Delta y}{L}; \qquad t = \frac{S}{L}; \qquad \lambda = \frac{q_0 L^3}{E I_0}, \tag{9}$$

and by substituting (7) in (1)–(5) we arrive to the following nonlinear system of equations describing nontrivial configuration of the column

$$\begin{aligned}
\dot{h} &= -\lambda(1 - \cos\theta);\\
\dot{v} &= \lambda \sin\theta;\\
\dot{m} &= -v\cos\theta + \left[-\lambda(1 - t) + h\right]\sin\theta;\\
\dot{\xi} &= 1 - \cos\theta;\\
\dot{\eta} &= \sin\theta;\\
\dot{\theta} &= \frac{m}{a^2},
\end{aligned} \tag{10}$$

where $\dot{(\cdot)} = \mathrm{d}(\cdot)/\mathrm{d}t$. The boundary conditions corresponding to (10) are

$$\xi(0) = 0; \qquad \eta(0) = 0; \qquad m(0) = 0;$$
$$\eta(1) = 0; \qquad m(1) = 0; \qquad h(1) = 0. \tag{11}$$

Note that the system (10)–(11) has the solution $h(t) = 0, \ldots, \theta(t) = 0$ for all values of λ. Next we linearize (10) to obtain

$$\dot{h} = 0;$$
$$\dot{v} = \lambda\theta;$$

$$\dot{m} = -v - \lambda(1-t)\theta;$$
$$\dot{\xi} = 0;$$
$$\dot{\eta} = \theta;$$
$$\dot{\theta} = \frac{m}{a^2}. \tag{12}$$

By using boundary conditions (11) in (12) we conclude that $h(t) = \xi(t) = 0$ and the rest of Eqs (12) could be reduced to

$$\ddot{m} + \frac{\lambda}{a^2}(1-t)m = 0, \tag{13}$$

subject to

$$m(0) = m(1) = 0. \tag{14}$$

The system (13)–(14) constitutes a linear spectral problem. For the case when $0 < a(t) < \infty$, $t \in (0, 1)$ the eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_i < \infty$, are simple and the only accumulation point is at infinity (see Churchill, 1969, p. 73). In accordance with this fact and Krasnoselskii's theorem (see Rabier, 1985) we have the following

PROPOSITION 1: *The bifurcation points of the system* (10)–(11) *are of the form* $(0, \lambda_n)$ *where* λ_n *are eigenvalues of the system* (13)–(14).

The column of greatest efficiency is the one that has minimal volume for fixed lowest eigenvalue λ_1 of (13)–(14). Thus, for the column of the greatest efficiency we have to minimize

$$J = \int_0^1 a(t)\,\mathrm{d}t, \tag{15}$$

subject to

$$\ddot{m} + \frac{\lambda_1}{a^2}(1-t)m = 0, \tag{16}$$

with boundary conditions (14). In (16) we consider λ_1 to be known. J in (15) has meaning of the dimensionless volume of the column.

3. Optimization problem and its solution

To determine the optimal shape of the column, i.e., $a(t)$ such that J is minimal, we shall use the Pontryagin's maximum principle (Sage and White, 1977). Let us rewrite optimization problem (15)–(16) as:

Find continuous function $u(t) = a(t)$, usually termed control, such that $0 < u(t) < \infty$, $t \in (0, 1)$, for which the optimality criterion

$$J = \int_0^1 u(t)\,\mathrm{d}t, \tag{17}$$

attains minimum value. The governing differential equations are

$$\dot{x}_1 = x_2; \qquad \dot{x}_2 = -\frac{\lambda_1}{u^2}(1-t)x_1, \tag{18}$$

subject to

$$x_1(0) = x_1(1) = 0. \tag{19}$$

The optimization problem just stated could be modified in many ways. For example, we could allow the piecewise continuous controls satisfying inequality type of constraints. The Pontryagin's maximum principle could also be applied to these types of problems (see Atanackovic, 1997, p. 217), but we shall not be concerned with them.

For the system (17)–(18) the Hamiltonian function $\mathcal{H}$ could be easily constructed as

$$\mathcal{H} = u + p_1 x_2 - p_2 \frac{\lambda_1}{u^2}(1-t)x_1, \tag{20}$$

where costate variables p_1 and p_2 have to satisfy the following system of differential equations

$$\begin{aligned} \dot{p}_1 &= -\frac{\partial \mathcal{H}}{\partial x_1} = \frac{\lambda_1}{u^2}(1-t)p_2; \\ \dot{p}_2 &= -\frac{\partial \mathcal{H}}{\partial x_2} = -p_1, \end{aligned} \tag{21}$$

subject to

$$p_2(0) = p_2(1) = 0. \tag{22}$$

The optimality condition $\min_u \mathcal{H}(t, x_1, x_2, p_1, p_2, u)$ leads to

$$\frac{\partial \mathcal{H}}{\partial u} = 1 + \frac{2\lambda_1}{u^3}(1-t)x_1 p_2 = 0. \tag{23}$$

By solving (23) for u we obtain

$$u = -\left[2\lambda_1(1-t)x_1 p_2\right]^{1/3}. \tag{24}$$

Note that the boundary value problems (18)–(19) and (21)–(22) are not coupled. Moreover, their solutions are related by the following equations

$$p_1(t) = x_2(t); \qquad p_2(t) = -x_1(t). \tag{25}$$

Therefore the control variable $u(t)$ given by (24) becomes

$$u = \left[2\lambda_1(1-t)x_1^2\right]^{1/3}. \tag{26}$$

Note that by using (25) in (20) we have $(\partial^2\mathcal{H}/\partial u^2) > 0$ so that the necessary condition for minimum of $\mathcal{H}$ is satisfied. By substituting (26) into (18) we arrive at single differential equation

$$\ddot{x} + \bar{\lambda}(1-t)^{1/3}x^{-1/3} = 0, \tag{27}$$

subject to

$$x(0) = x(1) = 0. \tag{28}$$

In (27) and (28) we used $x(t) = x_1(t)$, $\bar{\lambda} = (\lambda_1/4)^{1/3}$. Note that the end points $t = 0$ and $t = 1$ are singular points of the problem (27)–(28). To examine the local behavior of the solution near end points we first transform the independent variable t to $\zeta = 1 - t$. Then (27) becomes

$$\ddot{x} + \bar{\lambda}\zeta^{1/3}x^{-1/3} = 0, \tag{29}$$

subject to

$$x(0) = 0; \qquad x(1) = 0. \tag{30}$$

In (29) we used again $\dot{(\cdot)} = \mathrm{d}(\cdot)/\mathrm{d}\zeta$. Suppose that the solution $x(\zeta)$ of (29)–(30) in the vicinity of $\zeta = 0$ behaves as

$$\begin{aligned} &x(\zeta) \sim C\zeta^{\alpha}\left[1 + a_1\zeta + a_2\zeta^2 + a_3\zeta^3 + a_4\zeta^4 + \cdots\right]; \\ &\quad \zeta \to 0, \end{aligned} \tag{31}$$

where C, α and a_i, $i = 1, 2, \ldots$ are constants. By substituting (31) in (29) we conclude that

$$\begin{aligned} &\alpha = 1; \\ &a_1 = -\frac{1}{2}\bar{\lambda}C^{-4/3}; \qquad a_2 = -\frac{1}{36}\bar{\lambda}^2C^{-8/3}; \\ &a_3 = -\frac{7}{1296}\bar{\lambda}^3C^{-4}; \qquad a_4 = -\frac{23}{15552}\bar{\lambda}^4C^{-16/3}. \end{aligned} \tag{32}$$

In (32) the constant C remains undetermined. It is related to the first derivative of x at $\zeta = 0$. Namely $C = \dot{x}(0)$. Note that from (29) and (32) it follows that

$$\lim_{\zeta \to 0} \ddot{x}(\zeta) = -\frac{\bar{\lambda}}{C^{1/3}}. \tag{33}$$

To solve (29) numerically we shall write the first order system of differential equations

$$\begin{aligned} &\dot{x}_1 = x_2; \\ &\dot{x}_2 = -\bar{\lambda}\zeta^{1/3}x_1^{-1/3}. \end{aligned} \tag{34}$$

Then, we choose $x_2(0) = C$ and solve (34) as an initial value problem. The boundary condition $x_1(1) = 0$ will be satisfied by the shooting method. In the first integration step the right hand side of $(34)_2$ will be equated with (33).

4. Variational principles for (27), (28)

In this section we shall formulate complementary variational principles (see Arthurs, 1980) for the boundary value problem (27)–(28). Let $\mathbf{X}$ be the following space

$$\mathbf{X} = \left\{X(t) : X(t) \in C^2(0,1);\, X(0) = X(1) = 0\right\}, \tag{35}$$

where $C^2(0, 1)$ is the space of continuous functions mapping $(0, 1)$ into reals R, having continuous first and second derivatives. Note that assumption (see Section 3) that $u(t)$ is continuous, together with (18), implies that $\ddot{x}(t)$ in (27) is a continuous function. Thus, the solution of (27)–(28) belongs to $\mathbf{X}$. Next we consider the variational problem

$$I(X) = \int_0^1 \left[\frac{1}{2}\dot{X}^2 - \frac{3}{2}\bar{\lambda}(1-t)^{1/3}X^{2/3} \right] \mathrm{d}t \to \text{ extr}, \tag{36}$$

for $X(t) \in \mathbf{X}$. It is easy to see that the minimizing element of (36), if it exists, satisfies (27)–(28). Therefore

$$\delta I(x, w) = 0, \tag{37}$$

where the "variation" w is $w = X - x$. The variational principle (37) is called primary variational principle. Calculating the second variation of (36) on $X = x$ we obtain

$$\delta^2 I(x, w) = \int_0^1 \left[\dot{w}^2 + \frac{1}{3}\bar{\lambda}(1-t)^{1/3}x^{-4/3}w^2 \right] \mathrm{d}t. \tag{38}$$

From (38) it follows that

$$\delta^2 I(x, w) \geqslant \|\dot{w}\|^2_{L_2}, \tag{39}$$

where $\|\dot{w}\|_{L_2} = (\int_0^1 \dot{w}^2 \, \mathrm{d}t)^{1/2}$. By combining (37) and (39) we conclude that the following statement holds:

PROPOSITION 2: *On the solution $x(t)$ of the boundary value problem* (27)–(28) *the functional* (36) *attains minimum*

$$\min_{X \in \mathbf{X}} I(X) = I(x). \tag{40}$$

Now we proceed to construct the dual variational principle. Let us denote by $F(t, X, \dot{X})$ the Lagrangian function of the variational principle (37) that is

$$F(t, X, \dot{X}) = \frac{1}{2}\dot{X}^2 - \frac{3}{2}\bar{\lambda}(1-t)^{1/3}X^{2/3}. \tag{41}$$

We get the canonical form of (27)–(28) by introducing the variable P as

$$P = \frac{\partial F}{\partial \dot{X}} = \dot{X}, \tag{42}$$

and the function $K(t, X, P)$, also called Hamiltonian, connected with F by Friedrichs transformation

$$K(t, X, P) = P\dot{X} - F = \frac{1}{2}P^2 + \frac{3}{2}\bar{\lambda}(1-t)^{1/3}X^{2/3}. \tag{43}$$

The system (27)–(28) then becomes

$$\dot{X} = \frac{\partial K}{\partial P} = P; \qquad \dot{P} = -\frac{\partial K}{\partial X} = -\bar{\lambda}(1-t)^{1/3}X^{-1/3}. \tag{44}$$

In order to obtain the dual variational principle we have first to solve (43) for F. Then, in so obtained expression we substitute X in terms of $\dot{P}$ from $(44)_2$ and use the resulting expression in (36). After partial integration and application of boundary conditions, we obtain

$$G(P) = I\big(P, X(P)\big) = -\int_0^1 \left[\frac{1}{2}P^2 + \frac{1}{2}\bar{\lambda}^3 \frac{1-t}{\dot{P}^2} \right] \mathrm{d}t. \tag{45}$$

Let $x(t)$ and $p(t)$ be the solution of canonical system (44). Then it is easy to show that the following dual variational principle holds

$$\delta G(p, n) = 0, \tag{46}$$

where the variation n is $n = P - p$ and P is an admissible function, i.e., $P \in \mathbf{Y} = \{P:\ P \in C^1(0,1);\ \int_0^1 [(1-t)/\dot{P}^2]\,\mathrm{d}t < \infty\}$. Note that elements of $\mathbf{Y}$ need not satisfy any boundary condition. However, in practical applications P is usually expressed as $P = \dot{X}$, where $X(t) \in \mathbf{X}$. Also from (45) it follows

$$\delta^2 G(p, n) = -\int_0^1 \left[n^2 + 3\bar{\lambda}^3 \frac{1-t}{\dot{P}^4} \dot{n}^2 \right] \mathrm{d}t \leqslant -\|n\|_{L_2}^2. \tag{47}$$

By combining (46) and (47) we have the following

PROPOSITION 3: *On* $p(t)$ *the functional* (45) *attains maximum*

$$\max_{P \in \mathbf{Y}} G(P) = G(p). \tag{48}$$

From the results of Propositions 2 and 3 the chain of inequalities could be derived

$$G(P) \leqslant G(p) = I(x) \leqslant I(X). \tag{49}$$

We shall use (36) and (45) to obtain and estimate error of an approximate solution of (27)–(28). This is achieved by using the inequality

$$\|w\|_{L_\infty} = \sup_{t \in (0,1)} |w(t)| \leqslant \frac{\|\dot{w}\|_{L_2}}{\sqrt{2}}, \tag{50}$$

valid for any $w(t)$ satisfying $w(0) = w(1) = 0$. From (49) and (50) we obtain

$$\|w\|_{L_\infty} \leqslant \left[I(X) - G(P) \right]^{1/2}. \tag{51}$$

5. Results

In this section we present results of numerical integration of the system (17)–(18). The value of λ_1 was taken as the lowest eigenvalue of the column with constant cross-section having unit volume. Precisely, we take $a(t) = 1$, $A_0 = \pi R_0^2$, $I_0 = \pi R_0^4/4$ in (7), where R_0 is the radius of the cross-section that we assume to be circular. Then, from (15) we conclude that $J = 1$. With these values λ_1 become $\lambda_1 = 18.956266$ (see Atanackovic, 1997).

Numerical solution of (17)–(18) is shown in *figure 2*. Note that in accordance with (24) the cross-section of the column (in our notation u) is zero at both ends of the column. By assuming that the optimal column is also of circular cross-section with the radius $r(t)$ we can take $u(t) = A(t)/A_0 = (r(t)/R_0)^2$. The radius $r(t)$ of the optimal column as a function of dimensionless arc-length is shown in *figure 3*.

When the volume of the optimal column is calculated from (17) we obtain

$$J_{\text{opt}} = \int_0^1 u(t)\,\mathrm{d}t = 0.81051. \tag{52}$$

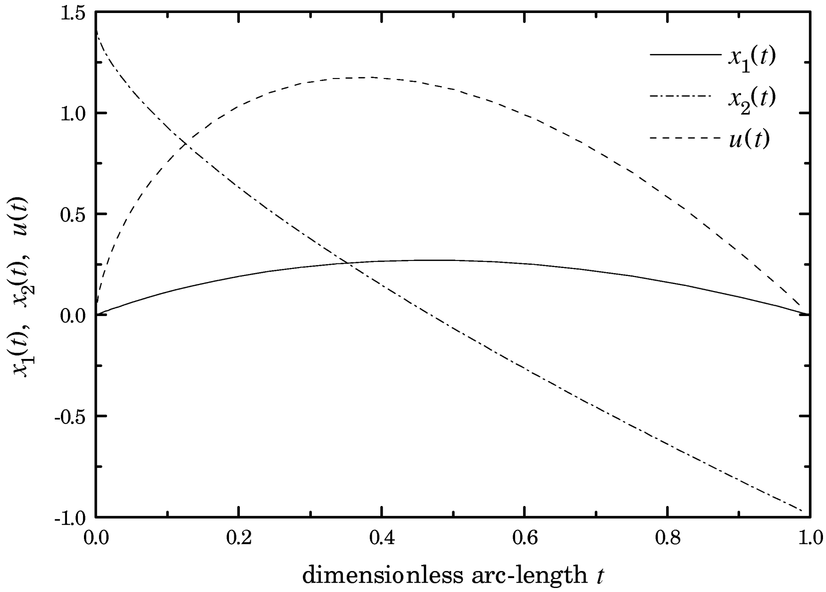

Figure 2. Numerical solution of (17)–(18).

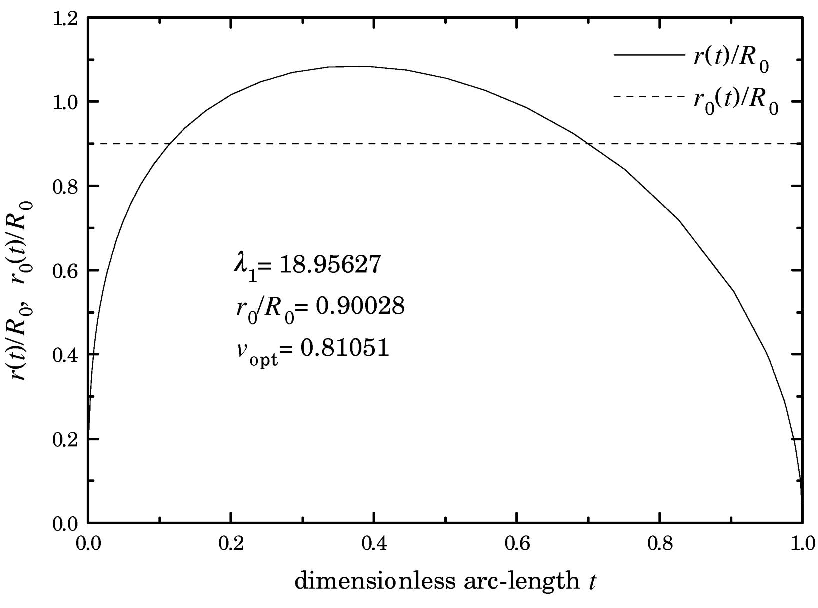

Figure 3. The shape of the optimal column.

It is interesting to compare the critical force of the column with constant cross-section having the same volume as the optimal column. The radius of the cross-section of this column is $r_0 = 0.90028R_0$. The corresponding dimensionless critical load is $\lambda_0 = 12.452807$.

We use now complementary variational principles, described in previous section, together with Ritz method (see Vujanovic and Jones, 1989) to determine an analytical approximate solution of the Eqs (27)–(28) for $\lambda_1 = 18.956266$, already used in numerical treatment. We assume the solution of the problem in the following

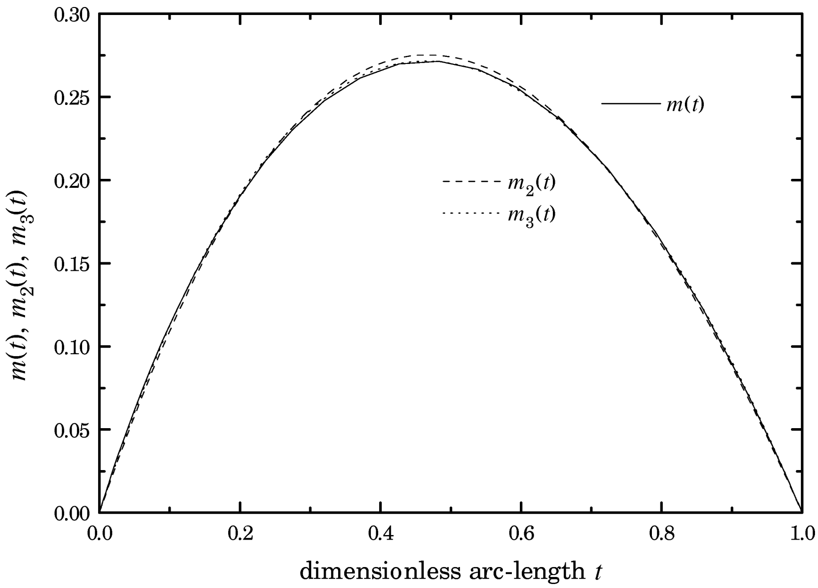

Figure 4. Exact and approximate solutions of (27).

form

$$X(C, C_1, C_2, t) = m_3(t) = Ct(1-t)\big(1 + C_1 t + C_2 t^2\big), \tag{53}$$

where C, C_1 and C_2 are constants to be determined. Since $x(t)$ in (27) represents the dimensionless moment (see (13)) we used this in writing (53). The index attached to m indicates the number of free constants that are to be determined. Note that $X(t)$ given by (53) is an admissible function since it satisfies the boundary conditions for all values of constants C, C_1 and C_2. By substituting (53) into (36) and minimizing with respect to C, C_1 and C_2 we obtain $C = 1.3036$, $C_1 = -0.4563$, $C_2 = 0.2293$ so that (53) and the corresponding value of functional I become

$$m_3(t) = 1.3036t(1-t)\big(1 - 0.4563t + 0.2293t^2\big); \qquad I(m_3) = -0.4052. \tag{54}$$

For the functional G given by (45) we take admissible function $P(t)$ in the form

$$P(D, D_1, D_2, t) = \dot{X}(D, D_1, D_2, t). \tag{55}$$

Substituting (55) in (45) and maximizing with respect to D, D_1 and D_2 we obtain

$$\begin{aligned} &P(D, D_1, D_2, t) = D\big(1 + 2D_1 t + 3D_2 t^2\big) - Dt\big(2 + 3D_1 t + 4D_2 t^2\big);\\ &\quad D = 1.3128,\ D_1 = -0.4646,\ D_2 = 0.2353;\\ &G(P) = -0.4088. \end{aligned} \tag{56}$$

From (55), (56) and (51) we obtain the estimate of the error of the solution (54) as

$$\|w\|_{L_\infty} \leqslant \big[I(X) - G(P)\big]^{1/2} = 0.06. \tag{57}$$

In *figure 4* we show the exact $(m(t))$ and two approximate solutions $m_3(t)$, given by (54), and $m_2(t)$ obtained by minimization of the functional I for the trial function with two unknown constants, i.e., $m_2(t) =$

$Ct(1-t)(1+C_1t)$. As it could be seen from *figure 4* by increasing the number of constants in trial function from two to three we obtained convergence of approximate solutions to the exact one. Once we know the approximate solution $m_3(t)$ we can determine the approximate cross-sectional area function of the column from (26).

6. Conclusions

In this paper we treated the problem of determining the optimal shape of a Pflüger column. Specifically we have done the following:

(1) We have derived the nonlinear system of equilibrium equations describing equilibrium configuration of a Pflüger column with variable cross-section.
(2) We have shown that under the condition that the cross-section does not vanish at any point inside $(0, 1)$, the eigenvalues of the linearized system of equations correspond to the bifurcation points of the full nonlinear system of equilibrium equations.
(3) By using Pontryagin's maximum principle we have derived Eq. (27) that determines the cross-sectional area function providing minimum volume (mass) of the column for given load.
(4) We have formulated complementary variational principles for Eq. (27). We have also demonstrated that these principles are extremal. They have been used to obtain an approximate solution and to estimate its error.
(5) Our analysis has shown that the cross-sectional area does not vanish in any point within the column's span so that our initial assumption about single modal optimization was correct.

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