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APPROXIMATE SOLUTIONS OF BIHARMONIC EQUATION USING WAVELET SUBSPACES

Nenad Teofanov
Institute of Mathematics, Faculty of Sciences, Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Srboljub Simić
Department of Mechanics, Faculty of Technical Sciences, Novi Sad
Trg Dositeja Obradovića 6, 21000 Novi Sad, Yugoslavia

Abstract

The boundary-value problem for the biharmonic equation $\nabla^4\Phi = 0$ is studied and an equivalent variational formulation of the problem is given. The sequence of wavelet subspaces is used to find the approximate solution of the variational problem by means of the Ritz Method. Theorems for the convergence of this method are presented.

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1. Introduction

Variational method plays a significant role in contemporary mathematics and physics. It unifies many diverse branches of science, leads to new theoretical results, and provides powerful methods of calculation. Many problems are usually posed first in the form of boundary-value problems for differential equations, integral equations, or more generally for operator equations. The inverse variational problem is to find an equivalent variational formulation of the previously formulated boundary-value problem, i.e. to define a functional on the suitably chosen space of functions so that it reaches the stationary value at the solution of the operator equation. If the solution to the inverse problem exists one can use many different methods for calculating approximate solutions of the problem which are based on the variational formulation.

In this paper it will be shown that a sequence of wavelet subspaces could be successfully used in order to find an approximate solution of a problem which arises in the theory of elasticity. The problem is originally posed as the boundary-value problem for a biharmonic equation and it will be transformed into an equivalent variational form. To be more precise, we are seeking for a weak solution of the problem. Finally, the Ritz method will be applied for the construction of the sequence of approximate solutions which converge to the exact solution of the problem. It will be also demonstrated that the spaces generated by wawelets represent an appropriate choice for the class of problems in consideration.

Let us recall some basic notions of the operator theory, calculus of variations and direct methods of approximation. Let $(\cdot,\cdot)$ and $\|\cdot\|$ be respectively the scalar product and the norm in the Hilbert space H. We consider the problem

$$Au = f \quad \text{on} \quad \Omega, \quad u = 0 \quad \text{on the boundary} \quad \partial\Omega, \tag{1}$$

where Ω is an interval in $\mathbf{R}$, and A is a linear, differential, positive–definite operator on the domain $\mathcal{D}_A$, which is dense in H.
Let us recall that an operator A is *symmetric* in $\mathcal{D}_A$ if, for all $u, v \in \mathcal{D}_A$ we have $(Au, v) = (u, Av)$. If for all $u \in \mathcal{D}_A$, $(Au, u) \geq 0$ and $(Au, u) = 0$ implies $u = 0$ we say that the symmetric operator A is *positive*. Finally, a positive operator A is *positive–definite* on its domain $\mathcal{D}_A$ iff there exists a constant $\mu > 0$ such that, for all $u \in \mathcal{D}_A$, we have $(Au, u) \geq \mu\|u\|^2$. Let H_A be a subspace of the space H, which consists of functions satisfying certain boundary conditions, given below. The weak solution of the operator equation (1) is the element of H_A which is a stationary point of the functional

$$J(u) = \frac{1}{2}(Au, u) - (f, u) = \frac{1}{2}\|u\|_A^2 - (f, u). \tag{2}$$

The inner product is in $L^2(\mathbf{R})$ sense, and $\|u\|_A$ is the norm of an element u in H_A.

We say that (2) is a variational formulation associated to the equation (1). It is not always possible to find a variational task equivalent to an arbitrary boundary value problem.

Notice that the stationary point is an element of the energy space H_A. The boundary conditions included in the definition of H_A are called *essential* boudary conditions. All the other boundary conditions of the given problem, called *natural* boundary conditions, are reproduced by the variational formulation itself, as a part of necessary conditions for the extremum.

Remark 1. If A is a differential operator of order $2m$, it is convenient to define the energy space H_A as a subspace of the Sobolev space $H^m(\mathbf{R})$.

The stationary point can be found by means of various methods. Let us now describe briefly the Ritz variational method, which we shall apply to a problem of linear theory of elasticity. The idea of the Ritz method is to reduce the infinite dimensional problem to the finite dimensional one. To that end, we consider a basis $\{\phi_1, \phi_2, ...\phi_N, ...\}$ of the separable energy space H_A. We define its finite dimensional subspaces S_N, $N \in \mathbf{N}$, as follows

$$S_N = \left\{ u(x) \in H_A \;\middle|\; u(x) = \sum_{j=1}^{N} \alpha_j \phi_j(x), \alpha_j \in \mathbf{R} \right\}.$$

The solution of the variational problem over the subspace S_N is a function $u_N \in S_N$ ($u_N = \sum_{j=1}^{N} c_j \phi_j$) such that the inequality $J(u_N) \leq J(v_N)$ holds for all $v_N \in S_N$. The coefficients c_j, which are called the *Ritz coefficients*, can be computed from the following system of equations

$$\frac{\partial J}{\partial c_i}(c_1, c_2, ..., c_N) = 0, \quad i = 1, 2, ..., N.$$

In our case this system is linear, and it has a unique solution ([R86]). In such a way we obtain a sequence of approximative solutions $\{u_N\}$. When $N \to \infty$, this sequence converges, in the weak sense, to the solution of the given operator equation (see [R86], page 262).

Notice that the choice of the basis $\{\phi_1, \phi_2, ...\phi_N, ...\}$ is closely related to the definition of the space H_A. In our case some important properties of wavelet bases will appear in the definition of H_A.

2. Definition of Wavelets

We present here only some basic facts on wavelets. For more details we reffer to [M90], [D92], [K94], [BF94], [M93].

We start with a *scaling function* (or father wavelet) $\phi(x)$ which has the properties

$$\left|\left(\frac{d}{dx}\right)^q \phi(x)\right| \leq C_m(1+|x|)^{-m}, \quad x \in \mathbf{R}, \ \forall q \in \mathbf{Z}, 0 \leq q \leq r, \quad \text{and all} \quad m \in \mathbf{N},$$

and $\int \phi(x)dx = 1$. In that case we say that ϕ is $r-$regular. In this paper we take $r \geq 2$. The *basic wavelet* (or mother wavelet) $\psi(x)$ is defined by

$$\hat{\psi}(\xi) = ((\hat{\phi}(\xi/2))^2 - (\hat{\phi}(\xi))^2)^{1/2} e^{-i\xi/2}. \tag{3}$$

(The Fourier transform is defined by $\hat{f}(\xi) = \dfrac{1}{2\pi}\displaystyle\int_{\mathbf{R}} f(x)e^{-ix\xi}dx$.) We immediately see that the wavelet ψ is of the same regularity as the scaling function ϕ.

Let us now put $\phi_k(x) = \phi(x-k)$, $\psi_{jk}(x) = 2^j\psi(2^j x - k)$, $k \in \mathbf{Z}$, $j \in \mathbf{N} \cup \{0\}$. The collection

$$\{\phi_k(x), \ \ \psi_{jk}(x), \ \ k \in \mathbf{Z}, \ \ j \in \mathbf{N} \cup \{0\}\}$$

is then an orthonormal basis of the space $L^2(\mathbf{R})$, and also an unconditional basis of the Sobolev spaces $H^s(\mathbf{R})$, $|s| \leq r$. Therefore, for $f(x) \in L^2(\mathbf{R})$, we have the following decomposition

$$f(x) = \sum_{k \in \mathbf{Z}} \beta_k \phi_k(x) + \sum_{j=0}^{\infty} \sum_{k \in \mathbf{Z}} \alpha_{jk} \psi_{jk}(x). \tag{4}$$

Let us consider the following partial sums:

$$\tilde{f}_n(x) = \sum_{k \in \mathbf{Z}} \beta_k \phi_k(x) + \sum_{j=0}^{n} \sum_{k \in \mathbf{Z}} \alpha_{jk} \psi_{jk}(x), \quad n \in \mathbf{N} \cup \{0\}; \tag{5}$$

$$f_N(x) = \sum_{k=-N}^{N} \beta_k \phi_k(x) + \sum_{j=0}^{N} \sum_{k=-N}^{N} \alpha_{jk} \psi_{jk}(x), \quad N \in \mathbf{N} \cup \{0\}. \tag{6}$$

In the next section we shall use the following assertions.

Theorem 1. ([M90] page 41) *Let ϕ be an r-regular scaling function, and ψ the wavelet defined by (3). If $f \in H^s(\mathbf{R})$, $|s| \leq r$, then the sequence of partial sums $\tilde{f}_n$ given by (5) of the series (4) converges to f in the norm of the space $H^s(\mathbf{R})$.*

An immediate consequence of the theorem is the following corollary.

Corollary 1. *Let ϕ be an r-regular scaling function and ψ the wavelet defined by (3). If $f \in H^s(\mathbf{R})$, $|s| \leq r$, then the sequence of partial sums f_N given by (6) of the series (4) converges to f in the norm of the space $H^s(\mathbf{R})$.*

3. An Application of the Wavelets

In this section we will describe the problem, transform it into a variational form, and apply the Ritz method in order to find an approximate solution.

We will consider a problem of the equilibrium of an elastic half-plane. It is a well-known problem of the theory of elasticity. One can find in classical textbooks (such as [S56]) that it can be reduced to quadratures but there is a limited number of problems with the solution given in closed form. Nevertheless, it may be of general interest to describe a procedure for constructing approximate solutions based on direct methods of variational calculus, which could be applied to a broad class of problems just mentioned. On the other hand, the following analysis is motivated by the question of applicability of wavelets in this framework.

Let us consider an elastic half–plane with a continuously distributed force $p(x_2)$ on its boundary. We assume that the material of the half–plane is linearly elastic, that is, its behaviour is determined by the Hook's law. Recall that the equilibrium of an elastic body is determined by Navier's equations. By introducing an auxiliary function, known as Airy's stress function $\Phi(x_1, x_2)$, $x_1 \geq 0$, $x_2 \in \mathbf{R}$, which is related to the stress field in the following manner

$$\sigma_{11}(x_1, x_2) = \frac{\partial^2 \Phi}{\partial x_2^2}, \quad \sigma_{22}(x_1, x_2) = \frac{\partial^2 \Phi}{\partial x_1^2}$$

$$\sigma_{12}(x_1, x_2) = \sigma_{21}(x_1, x_2) = -\frac{\partial^2 \Phi}{\partial x_1 \partial x_2}$$

the system of Navier's equations can be reduced to the biharmonic equation

$$\nabla^4\Phi = 0. \tag{7}$$

The boundary conditions are given by

$$\sigma_{11}(0,x_2) = -p(x_2) \quad \sigma_{12}(0,x_2) = 0 \tag{8}$$

$$\sigma_{ij}(x_1,x_2) \to 0 \text{ when } x_1, |x_2| \to \infty,\ i,j = 1,2.$$

The given problem has a solution, unique in the weak sense (see [A93]). The obtained mathematical model can be represented in an operator form

$$A\hat{\Phi}(x_1,\xi) = 0, \quad A = \frac{d^4}{dx_1^4} - 2\xi^2\frac{d^2}{dx_1^2} + \xi^2 \tag{9}$$

$$\hat{\Phi}(0,\xi) = \frac{\sqrt{2\pi}}{\xi^2}\hat{p}(\xi), \quad \hat{\Phi}'(0,\xi) = 0 \tag{10}$$

$$\hat{\Phi}(x_1,\xi) \to \infty \text{ when } x_1 \to \infty, \quad \hat{\Phi}'(x_1,\xi) \to \infty \text{ when } x_1 \to \infty, \tag{11}$$

where

$$\hat{\Phi}(x_1,\xi) = \frac{1}{2\pi}\int \Phi(x_1,x_2)e^{-ix_2\xi}dx_2.$$

As the operator A is positive ([R86], page 135) it is possible to associate an equivalent variational formulation to the given problem. The corresponding functional will be defined in such a way that the boundary conditions (10) appear as natural boundary conditions. The remaining boundary conditions (11) will be treated as essential conditions. The variational problem is of the form

$$J(\hat{\Phi}) = \frac{1}{2}B(\hat{\Phi},\hat{\Phi}) + K(\hat{\Phi}(0,\xi),\hat{\Phi}'(0,\xi),\hat{\Phi}''(0,\xi),\hat{\Phi}'''(0,\xi)) \to extr \tag{12}$$

$$B(u,v) = \int_0^\infty [\hat{u}''\hat{v}'' + 2\xi^2\hat{u}'\hat{v}' + \xi^4\hat{u}\hat{v}]dx_1 \quad \hat{u} = \hat{u}(x_1,\xi),\ \hat{v} = \hat{v}(x_1,\xi)$$

$$K = \left[\frac{\sqrt{2\pi}}{\xi^2}\hat{p}(\xi) - \hat{\Phi}(0,\xi)\right]\hat{\Phi}'''(0,\xi) + \hat{\Phi}''(0,\xi)\hat{\Phi}'(0,\xi).$$

The function $\hat{u}$ which is the stationary point of $J(\hat{u})$ with respect to the space H_A is the weak solution of the problem (9), (10),(11) at the same time. The necessary conditions for the extremum reproduce the differential equation (9) and the conditions (10). The essential boundary conditions (11) appear in the definition of the energy space:

$$H_A = \{f \in H^2(\mathbf{R}) \Big| \frac{d^\alpha}{dx_1^\alpha} f(x_1) \to 0, \quad |x_1| \to \infty; \;\; \alpha \le 2\}. \tag{13}$$

The norm of the space is defined by $\|u\|_A = (Au, u) = B(u, u)$.

By $\hat{v}(x_1)$ we denote a function of variable x_1 and parameter ξ. The scaling function $\phi_k(x) = \phi(x - k)$ and the wavelets $\psi_{jk}(x) = 2^{j/2}(2^j x - k)$, $j \ge 0$, $k \in \mathbf{Z}$, belong to the space H_A, they form an orthonormal basis of the space $L^2(\mathbf{R})$ and an unconditional basis of the Sobolev space $H^2(\mathbf{R})$. We define the finite dimensional subspaces S_N of the space H_A by

$$S_N = \Bigg\{ \hat{v}_N(x_1) \in H_A \;\Big| \hat{v}_N(x_1) = \sum_{k=-N}^{N} \beta_k \phi_k(x_1) + \sum_{j=0}^{N} \sum_{k=-N}^{N} \alpha_{jk} \psi_{jk}(x_1), \;\; \beta_k, \alpha_{jk} \in \mathbf{R} \Bigg\}.$$

Thus, the dimension of S_N is $(2N + 1)(N + 2)$. Our aim is to find the stationary point of the functional (12) on the S_N, which is equivalent to determine parameters β_k, α_{jk} (depending on ξ) from the necessary conditions of extrema, which are

$$\frac{\partial J}{\partial \beta_i} = 0, \qquad \frac{\partial J}{\partial \alpha_{in}} = 0.$$

In a developed form this system of linear algebraic equations can be written as

$$\sum_{k=-N}^{N} \beta_k A_{ik} + \sum_{j=1}^{N} \sum_{k=-N}^{N} \alpha_{jk} B_{ijk} + \frac{\sqrt{2\pi}}{\xi^2} \hat{p}(\xi) \phi_i'''(0) = 0 \tag{14}$$

$$\sum_{k=-N}^{N} \beta_k C_{nik} + \sum_{j=1}^{N} \sum_{k=-N}^{N} \alpha_{jk} D_{nijk} + \frac{\sqrt{2\pi}}{\xi^2} \hat{p}(\xi) \psi_{ni}'''(0) = 0$$

$i = -N, ..., N$, $n = 1, ..., N$ where

$$\begin{aligned}
A_{ik} &= B(\phi_i, \phi_k) - \phi_i(0)\phi_k'''(0) - \phi_k(0)\phi_i'''(0) + \phi_i''(0)\phi_k'(0) + \phi_k''(0)\phi_i'(0) \\
B_{ijk} &= B(\phi_i, \psi_{jk}) - \phi_i(0)\psi_{jk}'''(0) - \psi_{jk}(0)\phi_i'''(0) + \phi_i''(0)\psi_{jk}'(0) + \psi_{jk}''(0)\phi_i'(0) \\
C_{nik} &= B(\psi_{ni}, \phi_k) - \psi_{ni}(0)\phi_k'''(0) - \phi_k(0)\psi_{ni}'''(0) + \psi_{ni}''(0)\phi_k'(0) + \phi_k''(0)\psi_{ni}'(0) \\
D_{nijk} &= B(\psi_{ni}, \psi_{jk}) - \psi_{ni}(0)\psi_{jk}'''(0) - \psi_{jk}(0)\psi_{ni}'''(0) + \psi_{ni}''(0)\psi_{jk}'(0) + \psi_{jk}''(0)\psi_{ni}'(0).
\end{aligned}$$

Therefore, we have constructed the sequence of approximate solutions $\{\hat{u}_N\}$. This sequence converges to the exact solution $\hat{u} \in H^2(\mathbf{R})$ in the norm of the Sobolev space $H^2(\mathbf{R})$, as a consequence of Corollary 1.

Notice that, with increasing N, the already obtained coefficients in the system (14) remain unchanged. Finally, the sequence of functions Φ_N, $n \in \mathbf{N}$, obtained by applying the inverse Fourier transform to the approximate solutions $\hat{u}_N$ with respect to ξ ($\Phi_N(x_1, x_2) = \int \hat{u}_N(x_1, \xi)e^{ix_2\xi}d\xi$) converges to the function $\Phi(x_1, x_2)$, the weak solution of the problem (7) (8).

Therefore we have proved the following

Theorem 2. *Let us consider the problem* (9) *with the boundary conditions* (10) *and* (11). *The solution of variational problem* (12) *over the domain* H_A *is also the weak solution of the given problem. Further, the sequence of approximate solutions* $\hat{u}_N$, *obtained by the Ritz method with the basis of the spaces* S_N, *which consists of wavelets, converges to the exact solution of the problem* (9),(10),(11).

Remark 2. One can find the solution of biharmonic equation applying the Fourier transform with respect to x_2, and obtaining an ordinary differential equation with respect to x_1 [A93]. But, if we let $p(x_2)$ degenerate to a single point, the solution obtained in such a way explode. By solving the corresponding variational problem one can obtain the bounded approximate solutions.

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