FRACTIONAL DERIVATIVE VISCOELASTIC MODEL OF THE HAMSTRING MUSCLE GROUP

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Abstract: This paper deals with a new mathematical model of the hamstring muscle group. The proposed viscoelastic model includes fractional derivatives of stretching force and elongation as well as restrictions on the coefficients that follow from the second law of thermodynamics. On the basis of experimental data four coefficients of the model have been calculated by numerical procedure. Obtained results are verified by use of the Laplace transform method. The obtained muscle force model in time domain includes Mittag-Leffler-type function.

Keywords**:** Viscoelasticity, Fractional Derivative, Stress Relaxation, Hamstring

1. INTRODUCTION

In human anatomy, a *hamstring* refers to one of the tendons that makes up the borders of the space behind the knee. In modern anatomical contexts, however, they usually refer to the tendons of the semitendinosus, the semimembranosus, and biceps femoris. Those muscles together with corresponding tendons form the hamstring muscle group. The aim of this work is to propose a simplified viscoelastic fractional-derivative model of the hamstring muscle group.

During passive stretch the muscle-tendon unit is considered to have viscoelastic response, see Taylor *et al.* (1990). Viscoelastic material when stretched to a new constant length, analogous static stretching technique, will decline in tension over time, Magnusson *et al*. (1996). However, according to Catania and Sorrentino (2007), not all models which arise in applications are suitable for describing viscoelastic behaviour. When studying dynamics of that kind of materials, the selection of an appropriate rheological model is of great importance.

Because of the fact that stress is proportional to the zeroth derivative of strain for solids and to the first derivative of strain for fluids, it is natural to suppose that for materials that have properties of both solids and fluids (viscoelastic materials), stress may be proportional to the strain derivative of noninteger order α, where $0 < \alpha < 1$, Podlubny (1999). Namely, fractional calculus based constitutive models are a powerful extension of the standard integer calculus based models, that offer a new alternative for describing biomechanical properties of normal, diseased and healing tissues, see Doehring *et al.* (2005).

In what follows, by use of the generalized Zener model we intend to propose a new mathematical model of the hamstring muscle group. In doing so we plan to use the existing experimental data and the methods described in Podlubny (1999) and the Laplace transform as applied in Petrovic *et al.* (2005). The important property of the generalized Zener model is that it is able to predict behaviour of the viscoelastic material with significant accuracy, including only four parameters, see Petrovic *et al.* (2005). We expect that four constants included in the model, determined from the stress relaxation experiment, could give useful information on the state of the muscles.

2. METHODS

The simplified mechanical model of a hamstring muscle group and human leg is introduced and is similar to the one presented in Tozeren (2000). Hamstring muscle group is modelled by a viscoelastic rod (Fig. 1). Lengths *a*, *b*, *c* and *d* depend on observed sample so they are considered as known quantities. During the movement of the lower leg *OD* the length *L* of the muscle depends only on the angular position of the lower leg.

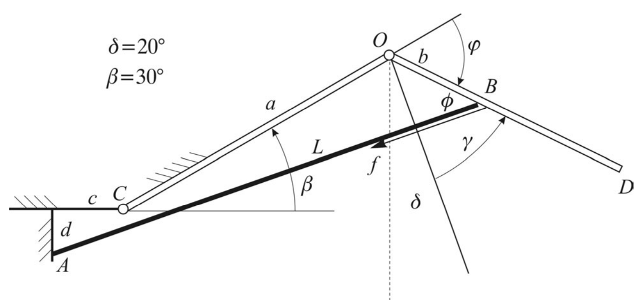

Fig. 1. System under consideration.

In Fig. 1 *a* denotes the upper leg length, *b* stands for the distance between the knee and the position where the hamstring is tied to the lower leg, while *c* and *d* describe the connection between the hamstring and a pelvic.

The length of the viscoelastic rod representing the muscle can be derived from the following equation

$$L(\gamma) = \frac{b + a\cos\left(\frac{5\pi}{9} - \gamma\right) + c\cos\left(\frac{5\pi}{9} - \gamma - \beta\right) - d\sin\left(\frac{5\pi}{9} - \gamma - \beta\right)}{\cos\phi} \tag{1}$$

where

$$tg\phi = \frac{a\sin\left(\frac{5\pi}{9} - \gamma\right) + c\sin\left(\frac{5\pi}{9} - \gamma - \beta\right) + d\cos\left(\frac{5\pi}{9} - \gamma - \beta\right)}{b + a\cos\left(\frac{5\pi}{9} - \gamma\right) + c\cos\left(\frac{5\pi}{9} - \gamma - \beta\right) - d\sin\left(\frac{5\pi}{9} - \gamma - \beta\right)}. \tag{2}$$

Elongation of the rod reads

$$x(\gamma) = L(\gamma) - L(0)\,. \tag{3}$$

The right choice of rheological model plays an important role in testing of viscoelastic materials. The model should enable good agreement with experimental data and, at the same time, contain as few parameters as possible. In recent studies it has been shown that in case of viscoelastic materials generalized Zener model, which comprises fractional derivatives, has more advantages than models which include integer order derivatives (Zener or Kelvin-Voight model), see Catania and Sorrentino (2007). For the generalized Zener model, which we use here, the constitutive equation has the following form:

$$f + \tau_{f\alpha} f^{(\alpha)} = E\left(x + \tau_{x\alpha} x^{(\alpha)}\right), \quad 0 < \alpha < 1, \tag{4}$$

where $f^{(\alpha)}$ and $x^{(\alpha)}$ are fractional time derivatives of a force and elongation given in the standard Reimann-Liouville form, see Gorenflo and Mainardi (2000)

$$\left[g(t)\right]^{(\alpha)} = \frac{d^\alpha}{dt^\alpha} g(t) = \frac{d}{dt}\left[\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{g(\tau)}{(t-\tau)^\alpha} d\tau\right]. \tag{5}$$

In (5) Γ stands for Euler Gamma function, $\tau_{f\alpha}$ and $\tau_{x\alpha}$ are constants of dimension $[\text{time}]^\alpha$, while the constant $E = E_\alpha A / L(0)$ contains the module of elasticity E_α, the cross-sectional area A of the viscoelastic rod and its initial length $L(0)$. Note that there exist fundamental restrictions on the coefficients of the model, that follow from the second law of thermodynamics, see Atanackovic (2002)

$$E > 0, \quad \tau_{f\alpha} > 0, \quad \tau_{x\alpha} > \tau_{f\alpha}\,. \tag{6}$$

We shall use this model to predict viscoelastic behavior of the hamstring muscle group during stress relaxation.

Four constants (α, E, $\tau_{f\alpha}$, $\tau_{x\alpha}$) describing the model will be determined on the basis of the experimental research on viscoelastic stress relaxation during static strech of hamstring muscles, see Magnusson *et al*. (1996). The change of angle γ and elongation x of the viscoelastic rod during the experiment is shown in Fig. 2, where a ramp-and-hold type of relaxation experiment could be recognized. Although more realistic than the classical stress relaxation experiment this type of experiments are not often encountered. During the phase 1 of the experiment the lower leg moves from the inital position defined by the fixed angle δ, at the angular speed of $\dot{\gamma} = 5$ °/s to its final position $\gamma = 80$ °, where it stays until the end of the experiment (phase 2), i.e. during the static phase (phase 2) angle γ remains stationary.

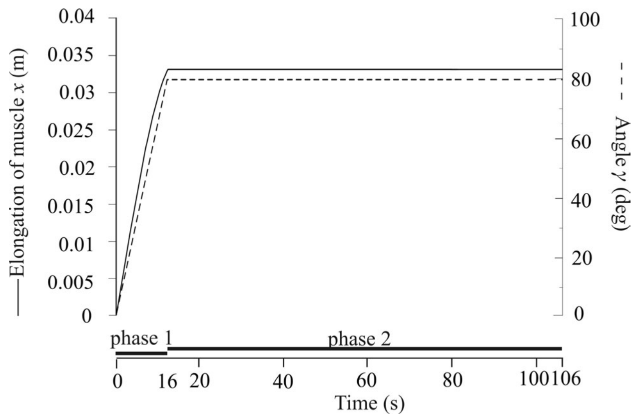

Fig. 2. Graphic representation of knee extension angle γ and elongation x of the viscoelastic rod.

During the experiment the passive torque M (Nm), which equals the moment of the force f in viscoelastic rod for the point O, is measured and represented by the following relation

$$M = f\, b \sin\phi\,. \tag{7}$$

In order to apply our model to experimental data, seven values of the passive torque were chosen from the stress relaxation curve presented in Fig. 4 of the paper of Magnusson *et al*. (1996). The appropriate muscle forces f_i (i=1,2,...,7) were calculated by (7) and used for the fitting procedure. All of the chosen experimental points were picked from the second phase of mentioned experiment because only data from the second phase were available.

There are several different methods which could be used for solving fractional-order differential equation (4), see Spasic and Charalambakis (2002). In this work two of them are applied for solving equation (4): numerical treatment ab initio and the Laplace transform method.

2.1 Numerical method

The numerical solution of differential equations of integer order has for a long time been a standard topic in numerical and computational mathematics. Here we deal with a simple but effective numerical method for solving fractional differential equations, which is far less advanced. This approach is based on the fact that for a wide class of functions, which appear in real physical and engineering applications, two definitions – *Reimann-Liouville* and *Grünwald-Letnikov* – are equivalent, Podlubny (1999).

For a time step h, $(t_m = m \cdot h,\ m = 0,1,2,...)$, we take the fractional derivative in the form

$$z^{(\psi)} = h^{-\psi} \sum_{j=0}^{m} \omega_j^{(\psi)} z_{m-j}, \tag{8}$$

where ψ is a real number $0<\psi<1$, and coefficients $\omega_j \, (j=0,...,m)$ are calculated by the recurrence relationships:

$$\omega_0^{(\psi)} = 1, \quad \omega_j^{(\psi)} = (1-\frac{\psi+1}{j}) \ \omega_{j-1}^{(\psi)}, \quad (j=1,\ 2,\ 3,...) \tag{9}$$

From (4) using (8) and (9) we get

$$f_m + \tau_{f\alpha} h^{-\alpha} \sum_{j=0}^{m} \omega_j^{(\alpha)} f_{m-j} = E\left(x_m + \tau_{x\alpha} h^{-\alpha} \sum_{j=0}^{m} \omega_j^{(\alpha)} x_{m-j} \right) \tag{10}$$

where the algorithm for obtaining the numerical solution for $m = 0,1,2,...$ reads

$$f_m = \frac{1}{1+\tau_{f\alpha} h^{-\alpha}} \left\{ E x_m \left(1+\tau_{x\alpha} h^{-\alpha}\right) + \right. \\ \left. + h^{-\alpha} \sum_{j=1}^{m} \left[\omega_j^{(\alpha)} \left(E\tau_{x\alpha} x_{m-j} - \tau_{f\alpha} f_{m-j} \right) \right] \right\} \tag{11}$$

For evaluating the muscle force, first the elongation of the viscoelastic rod, must be calculated from (3). Namely

$$\gamma_m = \begin{cases} \kappa \cdot m \cdot h, & m \cdot h < \overline{t} \\ \gamma_0, & m \cdot h \geq \overline{t} \end{cases} \tag{12}$$

where γ_0 stands for the knee extension angle γ in its final position, $\kappa = const$ represents the rate change of γ, and $\overline{t}$ is duration time of phase 1.

In order to calculate unknown parameters of the viscoelastic model muscle force should be defined as a function of time and these four parameters $f = f\left(t,\ \alpha,\ E,\ \tau_{x\alpha},\ \tau_{f\alpha}\right)$.

Than we computed the values of the parameters by the least squares method which means finding the minimum of the following function

$$\vartheta\left(t,\alpha,E,\tau_{x\alpha},\tau_{f\alpha}\right) = \sum_{i=0}^{6} \left[f\left(t_i,\alpha,E,\tau_{x\alpha},\tau_{f\alpha}\right) - f_{EXPi} \right]^2 \tag{13}$$

where f_{EXPi} are values of the muscle force in time instants t_i $(i = 1,2,...,7)$ which follows from the experiment and equation (7).

2.2 The Laplace transform method

We shall check the calculated values of the parameters by using the method of Laplace transformation.

By applying the Laplace transform $\mathcal{L}\left[g(t)\right] = \int_0^\infty e^{-st} g(t) dt = = G(s)$ to equation (4) we obtain

$$F(s) + \tau_{f\alpha} s^\alpha F(s) = E\left[X(s) + \tau_{x\alpha} s^\alpha X(s) \right] \tag{14}$$

Where we used the fact that

$$\mathcal{L}\left[z^{(\alpha)}(t) \right] = s^\alpha Z(s) - \left[\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{z(\tau)}{(t-\tau)^\alpha} d\tau \right]_{t=0}. \tag{15}$$

The term

$$\left[\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{z(\tau)}{(t-\tau)^\alpha} d\tau \right]_{t=0} \tag{16}$$

vanishes if $z(t)$ is bounded for $t \to 0^+$.

Equation (14) was solved for $F(s)$ to obtain

$$F(s) = EX(s) + E(\eta-1) \ \left\{ \frac{s^{\alpha-1}}{s^\alpha+\lambda} \left[sX(s) - X(0^+) \right] + \right. \\ \left. + \frac{s^{\alpha-1}}{s^\alpha+\lambda} X(0^+) \right\} \tag{17}$$

where $\eta = \frac{\tau_{x\alpha}}{\tau_{f\alpha}}$, $\lambda = \frac{1}{\tau_{f\alpha}}$. The term (17) is given in the form which is suitable for performing the inverse Laplace transformation. By inversion the muscle force in time domain is obtained, Gorenflo and Mainardi (2000),

$$f(t) = Ex(t) + E(\eta-1) \ \left[\int_0^\infty x'(t-\tau) e_\alpha(\tau;\ \lambda) d\tau + \right. \\ \left. + x(0^+) e_\alpha(t;\ \lambda) \right]. \tag{18}$$

In this term $e_\alpha(t;\ \lambda)$ stands for the Mittag-Leffler-type function, which is present whenever derivatives of fractional order in the constitutive equations of a viscoelastic body, are introduced, see Mainardi and Gorenflo (2000). Here we use the integral representation of the Mittag-Leffler function, given in Gorenflo and Mainardi (2000)

$$e_\alpha(t;\ \lambda) = \frac{1}{\pi} \int_0^\infty \frac{e^{-rt} \lambda r^{\alpha-1} \sin(\alpha\pi)}{r^{2\alpha} + 2\lambda r^\alpha \cos(\alpha\pi) + \lambda^2} dr, \\ 0<\alpha<1, \quad \lambda>0. \tag{19}$$

3. RESULTS

The values of the four unknown constants describing viscoelastic properties of the hamstring muscle group are computed by use of the suggested numerical procedure. We present results for the following case:

$x_0 = 3.314 \cdot 10^{-2}\,\text{m}$, $\overline{t} = 16\text{s}$, $\kappa = 8.7 \cdot 10^{-2}\,\text{s}^{-1}$, $a = 5 \cdot 10^{-1}\,\text{m}$,
$b = 3 \cdot 10^{-2}\,\text{m}$, $c = 8 \cdot 10^{-2}\,\text{m}$, $d = 6 \cdot 10^{-2}\,\text{m}$, $h = 0.1$, $N = 1062$

In time instants $t_i[\text{s}]$:

$t_1 = 16$, $t_2 = 26$, $t_3 = 36$, $t_4 = 46$, $t_5 = 56$, $t_6 = 86$, $t_7 = 106$,

the experiment together with (7) gives $f_i[\text{N}]$:

$f_1 = 1867.38$, $f_2 = 1689.49$, $f_3 = 1580.02$, $f_4 = 1518.45$,
$f_5 = 1484.24$, $f_6 = 1402.14$, $f_7 = 1347.41$.

Finding the minimum of the function from term (13) using the initial values: 0.395, 24000, 10.866, 3.04 for the unknown constants: α, E, $\tau_{x\alpha}$ and $\tau_{f\alpha}$ respectively, the following results are obtained:

$$\alpha = 0.517,\ E = 30070,\ \tau_{x\alpha} = 12.513,\ \tau_{f\alpha} = 4.948\,.$$

Fitting procedure was performed by Mathcad software. Restrictions (6) have not been implemented into the numerical adaptation process but the obtained results were in according to the restrictions. We also noticed that the values of unknown parameters almost do not change for variety of initial values:

The agreement between experimental results and the model is shown in Fig. 3.

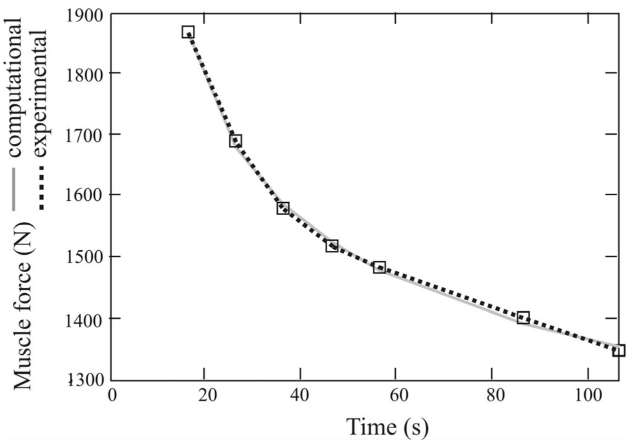

Fig. 3. Agreement between the stress relaxation curves for the sample of Magnusson *et al.* (1996) and fractional model (11) obtained by numerical treatment ab initio.

From Fig. 3 it can be observed that the difference between the experimental data and the model is negliable. The normalized root mean square error is less then 1.5%. If one use integer order model, for example the Prony (exponential) approximation, for the presented accuracy, more parameters are needed, see Pioletti and Rakotomanana (2000). The four parameter fractional model comprises history of deformation since fractional derivative represents non-local operator and thus is better choice than integer models based on local operators.

In addition, the muscle force given by (18) with the values of the constants determined by suggested numerical procedures is shown in Fig. 4. Note that both applied methods lead to results which are in a good agreement with experimetal data. It is very interesting that in this case the Laplace transform method found peak muscle point with the same accuracy that the rest of the relaxation values. We believe that nonsmoothness of elongation x near the maximum response can cause more divergence and that it depends on the number and the way of choosing experimental points, and that is why we picked seven experimental points in minimization procedure.

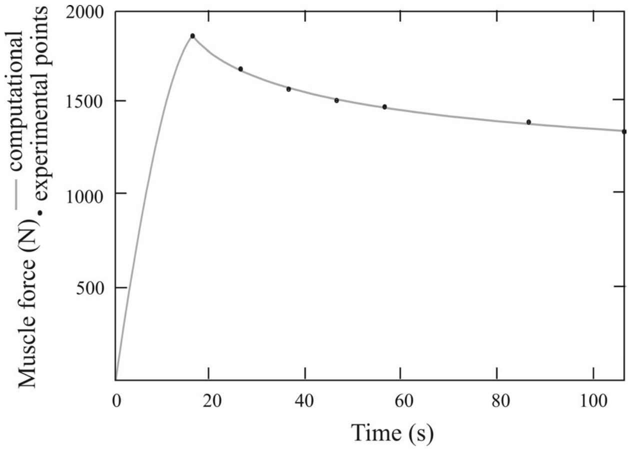

Fig. 4. Agreement between stress relaxation experiment data for the sample of Magnusson *et al*. (1996) and fractional model (18) obtained by the Laplace transform method.

5. DISCUSSION

In this paper we introduced a fractional-derivative viscoelastic model (4) for predicting mechanical behavior of a hamstring muscle group during stretching in ramp-and-hold stress relaxation experiment.

In order to obtain the muscle force as a function of time two methods are used leading to representations (11) and (18). Four constants of the viscoelastic model were calculated by numerical procedure, that follows equation (11), and by fitting through experimental data reported in Magnusson *et al.* (1996). We found good agreement between experimental results and theoretical predictions, which has also been confirmed by Laplace transform method and use of equation (18).

Finally, we comment on the fact that we use seven points in determining the four parameter model. Namely, four parameters could be obtained from the highly nonlinear function (18) by using only four experimental points. Then certain accuracy would be achieved in the chosen time domain. Increasing the number of experimental points leads to the less value of the mean square error, extends the time domain of good accuracy, but also needs more computing time. Thus, between four and much more than four, experimental points were chosen.

REFERENCES

Atanackovic, T.M., (2002). A modified Zener model of a viscoelastic body. In: *Continuum Mech. Thermodyn.* 14: 137-148. Springer-Verlag.

Catania, G., and S. Sorrentino. (2007). Analytical modelling and experimental identification of viscoelastic mechanical systems. In: *Advances in fractional calculus*: *Theoretical developments and applications in physics and engineering* (J. Sabatier *et al.* (Ed)), 403-416. Springer, Dordrecht, The Netherlands.

Doehring, T.C., A.D. Freed, E. O. Carew and I. Vesely, (2005). Fractional Order Viscoelasticity of the Aortic Valve Cusp: an Alternative to Quasilinear Viscoelasticity. In: *J. Biomech. Eng.* 127(4): 700-8. Elsevier

Gorenflo R., and F. Mainardi. (2000). Fractional Calculus: Integral and Differential Equations of Fractional Order. In: *Fractals and Fractional Calculus in Continuum Mechanics* (A. Carpinteri and F. Mainardi (Ed)), 223-276. Springer-Verlag, Wien and New York.

Mainardi, F., and R. Gorenflo. (2000). On Mittag-Leffler-type functions in fractional evolution processes. In: *Journal of Computational and Applied Mathematics*. 118: 283-299. Elsevier.

Magnusson, S.P., E.|B. Simonsen, P. Dyhre-Poulsen, P. Aagaard, T. Mohr and M. Kjaer. (1996). Viscoelastic stress relaxation during static stretch in human skeletal muscle in the absence of EMG activity. In: *Scand J Med Sci Sports*. 6: 323-328. Munksgaard, Denmark.

Petrovic, LJ.M. , D.T. Spasic and T.M. Atanackovic, (2005). On a mathematical model of a human root dentin. In: *Dental materials*. 21: 125-128. Elsevier.

Pioletti, D.P. and L.R. Rakotomanana, (2000). Non-linear viscoelastic laws for soft biological tissues. In: *Eur. J. Mech. A/Solids*. 19: 749-759. Elsevier.

Podlubny, I. (1999). *Fractional differential equations*. Academic Press, London.

Spasic, D.T, and N.C. Charalambakis. (2002). Forced vibrations with fractional type of dissipation. In: *Proceedings of the International Conference on Nonsmooth/Nonconvex Mechanics with Application in Engineering*, 323-330. Thessaloniki.

Taylor, D.C., J.D. Dalton, Jr., A.V. Seaber and W.E.J. GarrettHarris. (1990). Viscoelastic properties of muscle tendon units. The biomechanical effect of stretching. In: *Am J Sports*, 18: 300-309.

Tozeren, A., (2000). *Human body dynamics: Classical mechanics and human movement*. Springer-Verlag, New York.