A NOTE ON AIRPLANE LANDING PROBLEM

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Abstract: The paper treats a motion of an airplane landing on a straight line and stretching a weightless viscoelastic fiber whose ends are anchored at points a given distance from the line. The constitutive model of the viscoelastic fiber comprises fractional derivatives of stress and strain and the restrictions on the coefficients that follow from Clausius Duhem inequality. The dynamics of the problem may be represented by a single integral equation involving Mittag-Leffler-type function. The existence of the solution will be ensured by the Contraction Mapping Principle and will be obtained numerically by use of the first-order fractional difference approximation.

Keywords: Nonlinear Fractional Differential Equations, Airplane Landing

1. INTRODUCTION

The new tendency in engineering favors the design of slender structures incorporating new high performance materials. The use of such structures requires thorough knowledge of their physical properties, especially the study of viscoelastic response. This raises the problem of coupling geometric nonlinearity with either linear or nonlinear constitutive equations. According to Bagley (1989) it seems that a generalized linear model of a viscoelastic body that contains fractional derivatives of stress and strain is capable of describing viscoelastic behavior of real materials in a more accurate way then nonlinear constitutive models with derivatives of integer order. Following that lines, one may pose a problem how to treat finite deformations coupled with so called standard fractional viscoelastic body. This will lead to nonlinear fractional differential equations.

As stated by Seredyńska and Hanyga (2000), papers on nonlinear fractional differential equations are rare. Namely, in their paper on nonlinear pendulum and the Duffing equation, with a Caputo fractional derivative term replacing the usual damping, striking differences between ordinary differential and fractional differential equations are shown.

The aim of this paper is to add one more example to the list of nonlinear fractional differential equations.

2. THE PROBLEM

Consider a motion of an airplane landing on a straight line and stretching a weightless viscoelastic fiber whose ends are anchored at points a given distance from the line, see Fig. 1. Roughly speaking the landing script could be as follows. At the time $t = 0$ the airplane of mass m, with velocity v_0, touches the flight deck and at the same moment, it touches the weightless viscoelastic fiber, of length $2h$, which was perpendicular to the line of landing. The stretching of the fibre will proceed until the airplane slows dawn. With very low velocity, say at $t = \bar{t}$, the airplane will release the fibre and, in order to stop, will use the classical brake. This last part is not the subject here.

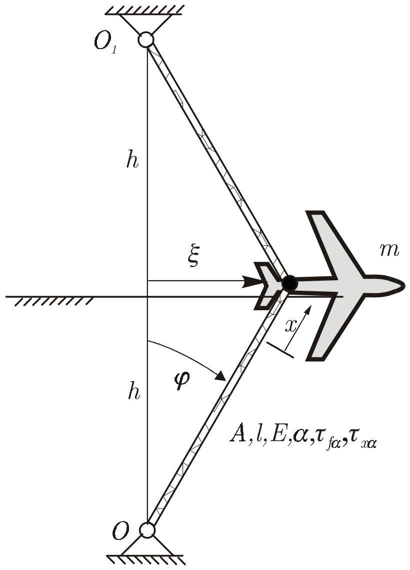

Fig. 1. System under consideration.

The differential equation of motion of an airplane and the initial conditions read

$$m\xi^{(2)} = -2f\sin\varphi,$$
$$\xi^{(1)}(0) = v_0,\ \xi(0) = 0,\ f(0) = 0, \tag{1}$$

where $(\cdot)^{(k)} = d^k(\cdot)/dt^k$ denotes the $k-$th derivative with respect to time t, and where $\xi = \xi(t)$ and $f = f(t)$ stand for the coordinate and the contact force between the airplane and the fibre. It should be noted that large values of ξ and φ (the angle describing the fibre deformation) are allowed.

The strain measures are often defined with special requirements in mind, see Atanackovic and Guran (2000). Let $x = x(t)$ be the half measure of the isothermal uniaxial deformation of the fibre. For simplicity, the relation between $f = f(t)$ and $x = x(t)$ (constitutive equation of the deformable fibre) may be taken in the following form

$$f + \tau_{f\alpha} f^{(\alpha)} = \frac{E_\alpha A}{h}\left(x + \tau_{x\alpha} x^{(\alpha)}\right), \tag{2}$$

where $0 < \alpha \leq 1$, A is the area of the fibre cross-section, E_α is the modulus of elasticity, $\tau_{f\alpha}$ and $\tau_{x\alpha}$ are the constants of dimension $[\text{time}]^\alpha$. In (2), for $0 < \alpha < 1$, $(\cdot)^{(\alpha)}$ denotes the $\alpha - th$ derivative of a function $(\cdot)$ taken in the Riemann-Liouville form as $d^\alpha[g(t)]/dt^\alpha = g^{(\alpha)} = d[\Gamma^{-1}(1-\alpha)\int_0^t g(\xi)(t-\xi)^{-\alpha}d\xi]/dt$, where Γ denotes the Euler Gamma function. In the special case when $\alpha = 1$ equation (2) represents the standard model of linear viscoelastic solid with τ_{f1} and τ_{x1} known as the relaxation times. Note that there exists fundamental restrictions on the coefficients of the model, that follow from the second law of thermodynamics, (Atanackovic, 2002),

$$E_\alpha > 0,\ \tau_{f\alpha} > 0,\ \tau_{x\alpha} > \tau_{f\alpha}, \tag{3}$$

Further, it is assumed that $x(0) = 0$.

In the following the obvious geometrical relations $\sin\varphi = \xi/(h+x)$ and $\xi^2 + h^2 = (x+h)^2$ are going to be very useful. Namely, introducing the dimensionless quantities $\bar{\xi} = \xi/h$, $\bar{x} = x/h$, $\bar{t} = t\left[2E_\alpha A/(mh)\right]^{1/2}$, $\bar{\tau}_{x\alpha} = \tau_{x\alpha}\left[2E_\alpha A/(mh)\right]^{\alpha/2}$, $\bar{\tau}_{f\alpha} = \tau_{f\alpha}\left[2E_\alpha A/(mh)\right]^{\alpha/2}$, $\bar{f} = f/E_\alpha A$ and $\lambda = v_0\left[m/(2E_\alpha A h)\right]^{1/2}$, one gets the following system describing the airplane landing phase

$$\xi^{(2)} = -f\frac{\xi}{\sqrt{1+\xi^2}},$$
$$\xi^{(1)}(0) = \lambda,\ \xi(0) = 0,\ f(0) = 0, \tag{4}$$

with

$$f + \tau_{f\alpha} f^{(\alpha)} = x + \tau_{x\alpha} x^{(\alpha)}, \tag{5}$$

and

$$\xi^2 + 1 = (1+x)^2. \tag{6}$$

In equations (4) - (6) the bar was omitted and the derivatives are taken with respect to dimensionless time. Also, the restrictions $(3)_{2,3}$ in dimensionless form remain the same.

The main concern of this work is the solution of (4) - (6). Before one proceed to it two remarks should be made here. First, by differentiating (6) twice, variable ξ could be eliminated, i.e., eq. (5) is to be solved together with nonlinear equation

$$x^{(2)} - \frac{\left[x^{(1)}\right]^2}{x(1+x)(2+x)} + fx\frac{(2+x)}{(1+x)^2} = 0, \tag{7}$$

with

$$x(0) = 0,\ x^{(1)}(0) = 0,\ f(0) = 0, \tag{8}$$

but this form of the problem is not enough tractable, (note that $x^{(2)}(0) = \lambda^2 \neq 0$). Secondly, the constitutive equation (5), so called the modified Zener model, is good enough to describe viscoelastic behaviour for wide class of real materials, metals, geological strata, glass, polymers for vibration control, even human root dentin, see Pritz (1996) and Petrovic *et al.* (2004), for example. When dealing with (5), a special attention should be paid to thermodynamical restrictions that should be observed in determining parameters of the model from experimental results. However in some problems, despite the fact it violates thermodynamical constraint $\tau_{f\alpha} > 0$, the term $\tau_{f\alpha}$ is small enough and could be neglected, (see (Fenander, 1998), where $\tau_{f\alpha}$ reads 0.69×10^{-9} $\sec^{0.49}$). In such cases the problem (5), (7) reduces to single nonlinear fractional differential equation

$$x^{(2)} - \frac{\left[x^{(1)}\right]^2}{x(1+x)(2+x)} + \frac{\left(x^2 + \tau_{x\alpha} x x^{(\alpha)}\right)(2+x)}{(1+x)^2} = 0, \tag{9}$$

with initial conditions $(8)_{1,2}$.

3. THE SOLUTION

In order to solve the landing problem the Laplace transform method will be applied. It will be shown that the dynamics of the problem is governed by a single integral equation involving Mittag-Leffler-type function, whose solution is ensured by the Contraction Mapping Principle, (Hutson and Pym, 1980). Introducing $X{=}X\left(s\right){=}\mathcal{L}\{x\left(t\right)\}{=}\int_0^\infty e^{-st}x\left(t\right)dt$ and $F{=}F\left(s\right){=}\ \mathcal{L}\{f\left(t\right)\}{=}\int_0^\infty e^{-st}f\left(t\right)dt$, from (5) one gets

$$F=\frac{1+\tau_{x\alpha}s^{\alpha}}{1+\tau_{f\alpha}s^{\alpha}}X, \tag{10}$$

where the standard expression for the Laplace transform of $z^{(\alpha)}$ was used, that is, $\mathcal{L}\{z^{(\alpha)}\}{=}s^{\alpha}Z\text{-}\left[\left(\int_0^t z\left(\xi\right)d\xi/\left(t-\xi\right)^{\alpha}\right)\right]_{t=0}$, with $\mathcal{L}\{z\left(t\right)\}{=}Z{=}Z\left(s\right)$, and where the term in brackets vanishes since $\lim_{t\to 0^+} z\left(t\right)$ is bounded, see (Oldham and Spanier, 1974). The inversion of (10) yields the following relation between $f\left(t\right)$ and $x\left(t\right)$

$$f\left(t\right)=\frac{\tau_{x\alpha}}{\tau_{f\alpha}}x\left(t\right)+\frac{1}{\tau_{f\alpha}}\left(1-\frac{\tau_{x\alpha}}{\tau_{f\alpha}}\right)\times \int_0^t e_{\alpha,\alpha}\left(t-\xi,\frac{1}{\tau_{f\alpha}}\right)x\left(\xi\right)d\xi, \tag{11}$$

where $e_{\alpha,\beta}\left(t;\lambda\right)$ stands for the generalized Mittag-Leffler function $e_{\alpha,\beta}\left(t;\lambda\right)\ \equiv\ E_{\alpha,\beta}\left(\text{-}\lambda t^{\alpha}\right)/t^{1-\beta}$ with $E_{\alpha,\beta}\left(t\right)\ =\ \sum_{n=0}^{\infty}t^{n}/\Gamma\left(\alpha n+\beta\right)$, (Gorenflo and Mainardi, (1997).

Next the geometrical relation (6) will be used. Namely, substituting $x\ =\ \sqrt{\xi^2+1}-1$ in (11) and the obtained function $f\left(t\right)$ into (4), the landing problem reduces to the following initial data problem

$$\xi^{(2)}=-\frac{\xi}{\sqrt{1+\xi^2}}\left\{\frac{\tau_{x\alpha}}{\tau_{f\alpha}}\left(\sqrt{1+\xi^2}-1\right)+\frac{1}{\tau_{f\alpha}}\left(1-\frac{\tau_{x\alpha}}{\tau_{f\alpha}}\right)\times\int_0^t e_{\alpha,\alpha}(t-\rho,\frac{1}{\tau_{f\alpha}})\left[\sqrt{1+\xi^2(\rho)}-1\right]d\rho\right\},$$

$$\xi(0)=0,\ \xi^{(1)}(0)=\lambda. \tag{12}$$

Thus, ξ has to satisfy the following integral equation

$$\xi(t){=}\int_0^t\left\{\lambda\text{-}\int_0^s\frac{\xi(u)}{1{+}\xi^2(u)}\left[\frac{\tau_{x\alpha}}{\tau_{f\alpha}}\left(\sqrt{1{+}\xi^2(u)}\text{-}1\right)+\frac{1}{\tau_{f\alpha}}\left(1-\frac{\tau_{x\alpha}}{\tau_{f\alpha}}\right)\times\int_0^u e_{\alpha,\alpha}(u\text{-}\rho,\frac{1}{\tau_{f\alpha}})\left(\sqrt{1{+}\xi^2(\rho)}\text{-}1\right)d\rho\right]du\right\}ds$$

$$={\mathcal M}\left(\xi\left(t\right)\right). \tag{13}$$

With this preparation done, one may apply the argument of the fixed point theorem. Let the ξ be in Banach space $C^1((0,T))$, equipped with sup-norm, for some $T>0$. In the following, the sign $\|\cdot\|$ will be used instead of $\|\cdot\|_{L^\infty((0,T))}$. Let B be the unit ball with center in $(\xi(0),\xi^{(1)}(0))=(0,\lambda)$ in $C^1((0,T))$, and $B_T=B\times[0,T]$. Note that

$$|\xi(t)-(0,\lambda)|\leq\left|\int_0^t\frac{\tau_{x\alpha}}{\tau_{f\alpha}}+\frac{1}{\tau_{f\alpha}}\times\left|\int_0^T e_{\alpha,\alpha}(s-\rho,\frac{1}{\tau_{f\alpha}})d\rho\right|\xi(s)ds\right|. \tag{14}$$

By taking the supremum of (14) over the interval $[0,T]$, one can see that $\mathcal{M}\left(\xi\right)\in B_T$, if $\xi\in B_T$ for T small enough. The first derivative can be estimated in the same way. Again, for $T_0<T$ small enough, one can see that $\mathcal{M}$ is a contractive mapping. Thus there exist a local solution (in the interval $[0,T_0]$). The uniqueness of such a solution.could be shown as follows. Suppose that ξ and ξ_1 are two solutions. Then

$$|\xi(t)-\xi_1(t)|\leq\left|\int_0^t\frac{\tau_{x\alpha}}{\tau_{f\alpha}}+\frac{1}{\tau_{f\alpha}}\times\left|\int_0^T e_{\alpha,\alpha}(s-\rho,\frac{1}{\tau_{f\alpha}})d\rho\right||\xi(s)-\xi_1(s)|ds\right|. \tag{15}$$

Gronwall inequality then implies that $|\xi(t)-\xi_1(t)|\leq 0$, i.e. $\xi\equiv\xi_1$. It could be shown that the above solution is the global one, because T_0 does not depend on the initial data.

Finally by use of the first-order fractional difference approximation, the influence of the four constants describing the fiber properties on the landing track, could be examined. Introducing the time step h, ($t_m=mh$, $m=1,2,...$), and the standard difference approximations for the first and second derivatives, the fractional derivative $z_m^{(\alpha)}$ may be taken in the form $h^{-\alpha}\sum_{j=0}^{m}\omega_{j,\alpha}z_{m-j}$, with $\omega_{j,\alpha}$ calculated by the recurrence relationships $\omega_{0,\alpha} = 1$ and $\omega_{j,\alpha} = (1 - (\alpha + j)/j)\,\omega_{j-1,\alpha}$, $(j = 1, 2, 3, ..)$, see (Podlubny (1999)). Then, taking into account the geometrical relation (6), the discretization of (4) and (5) read

$$\xi_m = 2\xi_{m-1} - \xi_{m-2} - \frac{h^2 f_{m-1}\xi_{m-1}}{\sqrt{1+\xi_{m-1}}},$$
$$f_m = \left(\sqrt{\xi_m^2+1} - 1\right)\frac{h^\alpha + \tau_{x\alpha}}{h^\alpha + \tau_{f\alpha}} +$$
$$\frac{\sum_{j=1}^{m}\omega_{j,\alpha}\left[\tau_{x\alpha}\left(\sqrt{\xi_{m-j}^2+1} - 1\right) - \tau_{f\alpha}f_{m-j}\right]}{h^\alpha + \tau_{f\alpha}},$$
$$m = 3, 4, ... \tag{16}$$

with $\xi_0 = 0$, $f_0 = 0$, and ξ_1, ξ_2 given as a solution of

$$\xi_2 - 2\xi_1 + h^2\frac{\xi_1\left(\sqrt{\xi_1^2+1} - 1\right)}{\sqrt{1+\xi_1^2}}\frac{h^\alpha + \tau_{x\alpha}}{h^\alpha + \tau_{f\alpha}} = 0,$$
$$-\xi_2 + 4\xi_1 - 2\lambda h = 0, \tag{17}$$

with

$$f_1 = \left(\sqrt{\xi_1^2+1} - 1\right)\frac{h^\alpha + \tau_{x\alpha}}{h^\alpha + \tau_{f\alpha}}$$

and

$$f_2 = \left(\sqrt{\xi_2^2+1} - 1\right)\frac{h^\alpha + \tau_{x\alpha}}{h^\alpha + \tau_{f\alpha}} +$$
$$\omega_{1,\alpha}\frac{\left[\tau_{x\alpha}\left(\sqrt{\xi_1^2+1} - 1\right) - \tau_{f\alpha}f_1\right]}{h^\alpha + \tau_{f\alpha}}.$$

4. RESULTS

In order to illustrate the above results, the numerical solution for the values α, E_α, $\tau_{f\alpha}$ and $\tau_{x\alpha}$ taken from the paper of Fenander (1998), will be presented. Namely, for $\alpha = 0.23$, and dimensionless values $\tau_{f\alpha} = 0.004$, $\tau_{x\alpha} = 1.183$, $\lambda = 1$, the motion of the system is presented in Fig. 2.

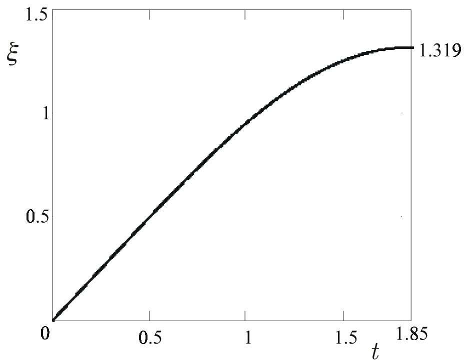

Fig. 2. Motion of the airplane for $\alpha = 0.23$, $\tau_{f\alpha} = 0.004$, $\tau_{x\alpha} = 1.183$ and $\lambda = 1$.

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