Stability of a compressed and twisted linearly-elastic rod

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1 Introduction

In this work the stability problem of a compressed and twisted rod is examined. The term rod denotes one-dimensional continuum. The words beam, column, shaft or bar could be used instead. The strategy to be adopted here bears on mathematical theory of elastic rods especially its four basic disciplines, shown below.

Mathematical theory of elastic rods			
Stability theory	Optimal control theory	Analytical mechanics	Singularity theory
A compressed and twisted rod			

The questions posed in the framework of stability theory and optimal control theory will be answered in Sections 1 and 2. However, the aim of the questions posed in Sections 3 and 4 is to make the rod problem open for applications of analytical mechanics and singularity theory.

In Section 1, the nonlinear differential equilibrium equations describing spatial buckled state of the rod will be derived. The equations will be obtained by use of fundamental geometric and mechanical principles, for physically fixed and in advance chosen object, and will be analyzed by use of the adjacent equilibrium method. This, usually called the Euler method states that the equilibrium configuration of a system, under given load and given boundary conditions, is stable if it is unique, and unstable if there is another, infinitesimally close, equilibrium configuration which could also be occupied . Hence, the stability problem reduces to the problem of qualitative analysis do equilibrium equations have one or several solutions. In that settlement the values of system parameters for which equilibrium equations have bifurcation points are the critical parameters defining the stability boundary. For some particular cases the critical values corresponding to the buckling load will be calculated. The existence of multiple solutions argument will be based on the standard Liapunov-Schmidt reduction procedure. Finally, for given post-buckling load, the equilibrium configuration will be determined numerically. Since the central result of this section concerns the effects of finite values of shear and extensional stiffness on the critical load for a compressed and twisted rod, its working title could be expressed as "Spatial buckling with shear and axial strain". The basic assumptions of the stability analysis are: the rod is made of a linearly elastic material with finite values of bending, extensional and shear stiffness; in the undeformed state the rod is straight and prismatic; its cross-section is arbitrary. Further, the ends of the rod are assumed to be attached to the supports by ideal spherical hinges and are free to rotate in any directions. Besides that, the compressing force and the twisting couple retain their initial directions during buckling what represents the nonconservative load. Finally, the use of the term - nonlinear equations - means that the finite deformations of the rod are assumed.

In Section 2 the problem of determining the compressed and twisted column of maximal efficiency will be formulated. Following the structural optimization theory, for a given load, the shape of the column of minimal weight will be determined. The working title of this part is "Optimal shape of a compressed and twisted rod against buckling". The problem will be solved numerically by use of Pontriagin's maximum principle. In doing so the linearized equilibrium equations of the previous section for the rod with circular cross-section and infinite values of shear and extensional stiffness will be used. The validity of that model follows from the bifurcation analysis performed in Section 1.

In the case of infinite values of shear and extensional stiffness, the differential equations describing a spatially deformed elastic rod in equilibrium are of the same form as the equations describing rotation of a rigid body about a fixed point. This result, known as Kirchhoff's analogy, will be used for tackling the rod problem by analytical mechanics approach. In Section 3, the structure of equilibrium equations for the case of compressed and twisted rod will be examined. Then, using the Nambu mechanics, the equilibrium equations of a twisted rod, will be derived. Finally, the possibility of obtaining the critical load for a compressed and twisted rod by reduction to algebraic bifurcation problem will be discussed.

In the last section the loss of stability for the ideal rod (i.e., the rod described in the first section) will be examined. By use of the Liapunov-Schmidt reduction procedure the bifurcation pattern will be identified. Also, the problem will be settled for application of singularity theory. Then, possibilities of introducing imperfections in shape and load will be analyzed. Namely, it is not easy to introduce imperfections in shape for finite values of shear and extensional stiffness. This is due to constitutive equations representing appropriate spatial generalization of the corresponding imperfect plane elastica models which are still difficult to find. Some relations of that kind will be given at the end of this section.

The rod under consideration is a typical engineering model for propeller shaft usually seen in shipping and aeroplane industries. Thus, some of the results presented here could be of interest in engineering practice.

2 Spatial buckling with shear and axial strain

2.1 Preliminaries

In this subsection some general remarks on generalizing the Bernoulli-Euler plane elastica theory as to take shear and compressibility into account, as well as theories of spatially deformed rods, are given. Also, a partial list of references on the stability problem of a compressed and twisted prismatic rod could be found.

The aim of any theory of rods or beams is to describe the deformed configuration of a slender three-dimensional body by a single curve and certain parameters recording material orientation relative to that curve. In doing so, the three-dimensional elastic constitutive law is replaced by expressions for resultant forces, moments and generalized moments in terms of extension, curvatures, torsion and remaining parameters. Each resulting theory must necessarily be approximate, although its accuracy should increase as the representative scale of distance along the axis of the rod increases relative to a typical diameter of the cross section, see Parker (1979, p. 361). In order to recognize some problems and possible expectations of spatial theories a short analysis of plane elastica theories is given.

The simplest model for the planar flexure of an elastic rod is the elastica, developed by Jas. Bernoulli, D. Bernoulli and Euler at the beginning of the eighteen century, see Antman and Rosenfeld (1978, p. 514). That theory neglects extensibility of the rod axis and shear stresses. The influence of the axial strain on the stability of a simply supported compressed elastic rod was first examined by Pflüger in the middle of this century, while the first work that generalizes the classical elastica, as to take shearing forces into account, goes back to Engesser who treated the problem in 1889, see Atanackovic (1989, p. 203) and Gjelsvik (1991, p. 1331). In the work of Pisanty & Tene (1973) the nonlinear equilibrium equations with both effects, derived from continuum mechanics were given, but constitutive equations were not considered. The constitutive relations that generalizes the Bernoulli-Euler elastica as to take effects of shear and compressibility were investigated by many authors. In what follows one possible classification of these theories is given. The following remarks are taken from the paper Atanackovic and Spasic (1992).

According to the way by which internal forces in an arbitrary cross section of the rod are decomposed, all generalizations could be classified in three different groups. In the first group the resultant force is decomposed into the direction of the rod axis and into the convected direction of the shared cross-section (which is, due to shear stresses, not perpendicular to the rod axis). In the literature this type of decomposition (into nonorthogonal directions) is usually called Engesser's approach. That approach, for example, was used in Schmidt and DaDeppo (1971), Pisanty and Tene (1973), Atanackovic and Spasic (1991), Spasic and Glavardanov (1993) etc. In the second group, that we together with Gjelsvik call Haringx's approach, the resultant force is decomposed into the convected direction of the shared cross section and into the direction normal to the sheared cross section. That approach was used in Reissner (1972), Antman and Rosenfeld (1978) and Goto *et al.*

(1990) for example. In his work Reissner posed the question that concerns whether the shearing angle should be connected with either the component of the resultant force in the convicted direction or the component of the resultant force in the direction of the normal to the deformed centerline. This leads us to the third group so called Timoshenko's approach in which the resultant force is decomposed into the direction of the rod axis and the direction orthogonal to the rod axis. Timoshenko's approach was formulated in Timoshenko and Gere (1961) and was used in Huddlestone (1972), DaDeppo and Schmidt (1972), Atanckovic *et al.* (1991), Atanackovic and Djukic (1992) etc.

In each of these groups another classification is possible. It is connected with the question should the relation between the shearing angle (defined as the angle between the convected direction and the direction of normal to the rod axis in the deformed state) and the shearing force be postulated in linear or nonlinear form. There are many references with possible answers. We pose two questions concerning the classification above. First, since similar relations are used to postulate different components what is the influence of the type of decomposition on the critical load of a given rod? The second question concerns the postbuckling behavior, see Goto *et al.* (1990, p. 386): is the deformation of the rod in postcritical region more accurately estimated by use of linear or nonlinear constitutive law?

In order to get the answers to these questions a heavy vertical column built up at the lower end was considered and the following models were tested:

1. Engesser's approach, linear version, see Schmidt and DaDeppo (1971)

$$\begin{aligned} M &= -EI\left(\dot{\vartheta}-\dot{\gamma}\right)\cos\gamma, \\ N_E &= EA\varepsilon, \\ Q_E &= \frac{GA}{k}\gamma, \end{aligned} \tag{1.1.1}$$

2. Haringx's approach, see Goto *et al.* (1990)

$$\begin{aligned} M &= -EI\left(\dot{\vartheta}-\dot{\gamma}\right), \\ N_H &= EA\left[(1+\varepsilon)\cos\gamma-1\right], \\ Q_H &= \frac{GA}{k}(1+\varepsilon)\sin\gamma, \end{aligned} \tag{1.1.2}$$

3. Timoshenko's approach, see DaDeppo and Schmidt (1972)

$$\begin{aligned} M &= -EI\left(\dot{\vartheta}-\dot{\gamma}\right), \\ N_T &= EA\varepsilon, \\ Q_T &= \frac{GA}{k}\gamma, \end{aligned} \tag{1.1.3}$$

4. Engesser's approach, nonlinear version, see Atanackovic and Spasic (1992)

$$
\begin{aligned}
M &= -EI(\dot{\vartheta} - \dot{\gamma}), \\
N_E &= EA\varepsilon, \\
Q_E &= \frac{GA}{k} \sin\gamma.
\end{aligned} \tag{1.1.4}
$$

In (1.1.1-4) M denotes the resultant couple in an arbitrary cross section of the rod, N and Q are the components of the resultant force (note that the index denotes the type of decomposition), EI, EA and GA are bending, extensional and shear rigidity respectively, k is the Timoshenko factor which depends on the shape of the cross-section and on the material, (see Renton (1991)), ϑ is the angle between the tangent to the rod axis and the vertical axis, γ is the shearing angle i.e., the angle between the sheared cross-section and the normal to the rod axis, and finally ε is the dilatation of the rod axis. We used dot to denote the derivative with respect to the arc length of the rod axis in the undeformed state.

The Bernoulli-Euler elastica is obtained from the above relations for $\varepsilon = 0$ and $\gamma = 0$. Note that although similar $(1.1.1)_{2,3}$ and $(1.1.3)_{2,3}$ are different because Q_E and Q_H are results of different decompositions. Also, although different $(1.1.1)_1$ and $(1.1.3)_1$ are derived from Hooke's law (for linearly elastic material i.e., the stress is linear function of the strain). In deriving $(1.1.4)_3$ the simple shear of finite amount model was used.

The models denoted by 1, 3, and 4 give the same result for the critical load of the heavy vertical rod. Although the linearized form of (1.1.2) is the same as (1.1.3) the predictions for the critical load obtained from the model 2 differs from that result. The agreement of all four models is possible only for particular values of extensional and shear rigidity, see Atanackovic and Spasic (1992).

The study of nonlinear behavior of the heavy rod according to the different constitutive models states that the predictions of maximal deflections for all models under considerations are different. For example, the nonlinear model (1.1.4) when compared with (1.1.2) gives larger maximum values for M, γ and ϑ, see Atanackovic and Spasic (1992). Nevertheless, each model has a chance to be more suitable for specific application (beam, column, helical spring) as suggested by Gjelsvik (1991).

The above results may be useful in the analysis of the spatial generalizations of the Bernoulli-Euler elastica theory.

A satisfactory theory for the large deflection of rods in space was established at the middle of nineteenth century. Namely, in 1859 Kirchhoff was the first who introduced constitutive equations for a linearly elastic spatially deformed rod. In his theory, called "Kirchhoff's kinetic analogue", he showed analogy between the equilibrium equations of the rod and the equations of motion of a rigid body with a fixed point. The validity of these constitutive relations which were derived for the case of a prismatic rod, which is straight in the undeformed state, was subject considered by Love in 1892 and Nikolai in 1916, see Filin *et al.* (1983, p. 37). Besides the incompressibility of the rod axis Kirchhoff's theory is related to so called the Bernoulli-Euler hypothesis, that is, the cross-section of the rod remains unchanged, plane and normal to the rod axis. Until early seventies, at authors knowledge, there were no theories which include effects of shearing forces and extensibility of the rod axis.

The first generalization of the classical Kirchhoff three-dimensional model of the elastic rod was given by Reissner (1973) where the influence of the extensibility and shear was considered. In this work the nonlinear equilibrium equations for spatially deformed rod were derived but constitutive equations in the explicit form were not considered. Only a hint how these equations could be postulated, by use of the principle of virtual work, was given, (see Reissner (1972) where the plane problem was analyzed in the same way).

Basically, constitutive relations of spatial theories are connected with two problems. The first group is related to recognitions and direct involvement of the effects seen in plane theories. This group is followed mainly by problems of geometrical nature. In the second group, the main problem is to introduce physically completely new effects motivated by results of three-dimensional elasticity theory. Namely, spatial deformations of the elastic rod are followed by changes of the rod cross-section, for example torsional-warping deformation.

Some theories that include effects of shear and compressibility do not involve changes of the rod cross-section directly but indirectly as it was the case in plane theories. Namely, in plane theories it is assumed that the cross-section remains plane while the effects of its changes are involved by multiplying the shear rigidity by so called the Timoshenko shear correction factor. That approach could be seen in Filin *et al.* (1983, p. 53) and Dupuis and Rousselet (1992, p. 479) where, roughly speaking, one can find immediate generalization of the linearized version of (1.1.2). The generalization of the plane elastica relation (1.1.2) without linearization could be found in works Simo *et al.* (1988), Eliseyev (1988) and Simo and Vu-Quoc (1991). Although constitutive equations could be postulated directly, the mentioned papers contain motivations for them. Namely, these equations were developed on the basis of the kinematical relations and considerations concerning the strain energy of the elastic rod. In addition we note that Simo and Vu-Quoc include torsional-warping deformation what is not the case with Eliseyev. However, by taking the influence of shear and extension on the resultant couple and by taking into account the influence of bending on the resultant force the work of Eliseyev pretends to generalize the relations obtained in the plane elastica theories for naturally curved rods. It is worth noting that for the initially straight and prismatic rod without considering torsional-warping deformation papers of Simo *et al.* (1988), Eliseyev (1988) and, Simo and Vu-Quoc (1991) lead to the same rod model. A somewhat different type of rod models could be found in the works of Kingsbury (1985), Iura and Hirashima (1985) and Goto *et al.* (1985). Referring to three approaches in plane elastica rod models mentioned above we note that in the works of Eliseyev and Simo & Vu-Quoc, the Haringx approach was recognized, while in the paper of Kingsbury the Timoshenko approach was used. Although the constitutive relations based on Engesser's approach for spatial buckling rod model are still unknown, as far as author's knowledge is concerned, probably it will be useful have them. The motivation for that assumption is based on Nanni's work published in 1971 where on the basis of the three-dimensional elasticity theory Engesser's approach was found out to be superior rod model compared to other approaches, see Ziegler (1982).

On the other hand theories that treat deformation of the cross-section start with the assumption of the deformation mode and then determine its intensity. The effects of the cross-section deformation become more complicated if in its undeformed state the rod is not straight and prismatic but twisted and curved (i.e., naturally curved or imperfect in geometrical sense). It is fair to say that still there is no general consistent constitutive theory of elastic rods which will include satisfactorily all the mentioned effects (see Filin *et al.* (1983, p.38).

A partial list of references in English on the constitutive theory, which can be used in stability analysis of imperfect rods, is presented in the paper of Rosen (1991). As far as the references in Russian are considered the analogue list could be found in the already mentioned monograph of Filin *et al* (1983). Also therein, the constitutive relations for small deformations i.e., for the geometrically linear theory of Kirchhoff and Clebsh with shear and axial strain for the case of naturally curved but not twisted rod, could be found. Because of their relatively high complexity the technical application of that relations is rather difficult. In author's opinion for geometrically non-linear theory the constitutive relations for naturally curved and twisted rod are well established only for infinite values of the extensional and shear rigidity. In the opposite case, when those rigidities are finite, there are many solutions, but solutions which generalize plane relations directly are still rare. Almost all the solutions that connect the load, elastic properties of the rod material and measures of deformation are motivated by the linear theory of elasticity, which can treat the rod problem by itself. Despite the fact that it is constrained on small deformations and small displacement linear elasticity theory uses very complicated mathematical apparatus, see Berdichevskii and Starosel'skii (1983,1985). Since the primary task of this thesis is the stability problem of initially straight and prismatic rod, considerations on geometrically imperfect rods will be postponed until Subsection 4.4.

Finally, in order to motivate the strategy that follows, the stability problem for a compressed and twisted rod, analyzed in the sense of the classical Kirchhoff theory, is briefly reviewed.

The first systematic experiments on the stability of compressed rods in equilibrium date back to 1729 and works of Musschenbroek. The problem of determining the shape of an elastic column when loaded with its own weight and with concentrated force at the top was formulated in 1728 by D. Bernoulli, see Atanackovic (1986, p. 361). The first theoretical results on stability analysis, published between 1744 and 1780 belong to Euler, see Filin (1981, p.286). Euler used the static method to determine the stability boundary for an axially loaded slender rod for several types of boundary conditions. In 1883 Greenhill studied the buckling of the rod with circular cross-section under terminal thrust and torsion. On the basic of linearized equilibrium equations Grammel (1923) derived the equation whose roots determine the critical load of the compressed and twisted rod with arbitrary cross-section. Both works assumed that the compressing force and the twisting couple do not change their direction during buckling. Nikolai (1928) examined different types of boundary conditions and analyzed what was implicitly assumed in both Greenhill's and Grammel's work, for example that the rod was hinged at both ends. He concluded that for a specific load the only equilibrium configuration is the trivial one and in such cases the dynamic equations should be used in stability analysis. The theory introduced by Cosserat in 1907 and improved by Ericksen and Truesdell in 1958 significantly contribute to the removal of all implicit assumptions in stability problems. Namely, in these works Kirchhoff's theory was embedded in a more general theory of one dimensional media in which kinematics, mechanics and constitutive relations are carefully based on fundamental principles, see Antman (1972, p. 665). Following the lines of Nikolai's work Ziegler in 1951 made more accurate examinations of the rod under terminal twist. He showed that the axial torque represents nonconservative load. Trosh (1952) confirmed the result of Greenhill by use of the dynamical method. These works show that the boundary of application of the static (Euler's) method of stability analysis does not coincide with the boundary between conservative and nonconservative problems. The mechanism how the twisting couple could be applied to the elastic rod was explained by Beck (1955). Beck analyzed five different types of supports and examined in what cases the conservative torsion occurs. Especially, he examined the critical value of twisting couple for each of the five cases and for the rods with both equal and different bending rigidities. Kovari (1969) examined compressed at both ends rigidly supported column and concluded that behind the Euler critical load there is a critical load when lateral buckling occurs. The analysis performed by Kovari was done on nonlinear equations. Besides that, he showed how the advantages of closed form solutions are lessened by the need to solve complicated transcendental equations involving elliptic functions and integrals. Such equations are generated by all except the most innocuous of boundary conditions. That these equations must be solved numerically suggests that a far more efficient process would be to give the original boundary value problem a numerical treatment *ab initio*.

The works of Antman (1972) represent the beginning of the mathematical theory of nonlinearly elastic rods in which the qualitative analysis of the stability problems was based on the bifurcation theory. Zachmann (1979) used the implicit function theorem in order to show the existence of different equilibrium configurations for the case of linearly elastic compressed and twisted rod with circular cross-section and for a conservative load. Antman & Kenney (1981) for a material described by general nonlinear constitutive equations which include shear and compressibility studied in detail the stability problem of a compressed and twisted rod. From that theory the Kirchhoff theory is obtained by imposing constrains which remove compressibility of the rod axis and shear strains and by replacing the general constitutive equations by the set of well-known linear equations. The constitutive equations of Antman and Kenney do not connect the load, the elastic properties of the material and the measure of deformation explicitly and thus could not be used for the determination of the stability boundary. Beda (1990), for the same problem as Zachmann, used Liapunov-Schmidt reduction and by use of the bifurcation equation examined the loss of stability. He showed that different load parameters correspond to different bifurcation patterns. Spasic (1991) made three additions to the problem posed by Grammel in 1923. First, the stability analysis was performed on nonlinear equations, secondly, the critical load was determined numerically and finally, the postcritical shape of the rod was obtained by numerical integration.

In what follows the problem posed by Grammel in 1923 will be analyzed by use of the rod model proposed by Eliseyev in 1988.

2.2 Rod description, physical and mathematical model

Consider a space curve, say L, of finite but variable length, and an unchangeable plane figure Q whose diameter is much smaller than the length of L. Suppose the center of the figure Q is connected to the curve L by a spherical hinge (a ball in a socket connection) which is able to move along the curve. We define the elastic rod with axis L and cross-section Q to be solid body which occupies the geometrical space ousted by the figure Q during the motion from one side of the curve to another. All quantities which describe that motion are assumed to be sufficiently smooth functions of the arc length of the curve.

We assume the rod is hinged at either end. One hinge, say A, is assumed to be fixed whereas the other one, say B, is free to move along the axis defined by the points A and B. We assume the rod is loaded by the compressing force P, acting at B, having the action line along AB, and by two twisting couples of equal intensity and opposite directions, say W, acting at A and B respectively, and having the action line along AB. It is assumed that both the force and the couples could not change their directions during buckling.

In order to investigate the stability of equilibrium of the rod according to the Euler method[1] one need three categories: the undeformed state, the trivial configuration and the nontrivial configuration. In the undeformed state the rod is assumed to be straight and prismatic i.e., the axis of the rod coincides with the straight line between the points A and B, say of length L. This state corresponds to zero values of P and W and a translation of the figure Q from one side to another. The trivial configuration appears for relatively small values of the force and the couple, say $P > 0$ and $W > 0$. In that equilibrium state the axis of the rod remains straight although of smaller length than in the undeformed state (the points A and B are at the distance less than L now). To pass all the points of the rod in this configuration the figure Q should rotate about the fixed axis AB and moves along it. Namely, Q remains normal to the axis of rotation which coincides with AB. Finally, if the force P and the couple W become greater (for relatively sufficiently small amount) than certain critical values, say $P > P_{cr}$, $W > W_{cr}$, besides the trivial one a nontrivial equilibrium configuration could appear. The nontrivial configuration, which is very close to the trivial one, corresponds to the equilibrium state in which the rod axis is spatial curve and the model for motion of Q is general motion of a rigid body. This

[1] also called static or method of adjacent equilibrium,

is a spatial buckled state of the elastic rod. The length of the rod axis is not necessarily L. This position corresponds to the nontrivial or postcritical configuration of the system. Namely, if the rod axis went to far from given allowed area the rod, and also the system whose part the rod is, will lose its function. According to the Euler criterion the values of P_{cr} and W_{cr} in whose neighborhood besides the trivial one also the nontrivial configuration could appear will be examined.

As far as the physical model of the rod is considered it is assumed that the rod is made of an isotropic linearly-elastic material. The bending and torsional rigidities for principal axes of the cross-section Q are EI, EJ and GJ respectively. The fact that the figure Q remains plane and unchanged means the involvement of the Bernoulli hypothesis. The effects of compressibility are recognized by the variable length of the space curve under consideration. The effects of shear are taken into account by means of the connection between Q and L. Namely, the spherical hinge does not involve any constrains on the angle between the tangent to the rod axis and the normal to the plane figure Q representing the plane of the cross-section. In the classical Bernoulli-Euler elastica that angle is always zero. The fact that the compressing force and the twisting couple remain in their initial direction during buckling means that the nonconservative load is under consideration, see Beck (1955). Namely, in such a case the work of the couple W depends on the movement of the tangent to the rod axis during buckling, see Timoshenko and Gere (1963, p. 156).

At last here is the mathematical model needed for analysis of the position and load of the rod. The position of the rod is determined by the shape of the rod axis and the orientation of the cross-section Q. Let S be a coordinate that represents the arc length of the space curve L in an undeformed state, what corresponds to the Lagrange description. In spatial buckling problems both the rod axis and the orientation of the rod cross-section are determined by three scalar functions of the coordinate S each. The same is valid for the resultant force and the resultant couple in an arbitrary cross-section of the rod. Therefore, twelve functions of S describing the deformed state of the rod are to be defined on the finite interval $[0, L]$. As it was stated all the functions are assumed to be smooth enough what will made the analysis of the problem more tractable.

2.3 Geometrical relations

In this subsection the functions describing the rod position are examined.

The rod axis is in general case a spatial curve. In all possible positions of the rod the rod axis will be represented in the fixed coordinate system, say $Oxyz$, with $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ as the corresponding unit vectors, with the origin at the point $A \equiv O$ and the z axis which coincide with the rod axis in the undeformed state (AB direction). The rod axis is then represented by means of the vector $\mathbf{r}(S) = x(S)\,\mathbf{i} + y(S)\,\mathbf{j} + z(S)\,\mathbf{k}$.

Denote by s the arc length of the rod axis in the deformed state. An element of the rod axis whose length in the undeformed state is dS in the deformed state has the length

ds. The strain of the rod (central) axis is defined as

$$\varepsilon = \frac{ds - dS}{dS}. \tag{1.3.1}$$

The deformation measure of the rod axis will be introduced in Subsection 1.5.

In each point of the rod axis i.e., at the centre of the plane figure Q, say C, the coordinate system, fixed to Q, say $C\xi\eta\zeta$, with $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ as the corresponding unit vectors, is attached. The $C\zeta$ axis is oriented along the normal to Q and the axes $C\xi$ and $C\eta$ coincide with the principal axis of the cross-section Q. The motion of this coordinate system describes the motion of Q from one end of the rod axis to another and thus describes orientation of the cross-section in the deformed state. With respect to the fixed coordinate system $Oxyz$ the position of the coordinate system $C\xi\eta\zeta$ at each point of the rod axis is described by the three spherical angles of the Euler type , say $\psi = \psi(S)$, $\vartheta = \vartheta(S)$ and $\varphi = \varphi(S)$. The relations between the systems $Oxyz$ and $C\xi\eta\zeta$ will be given in Subsection 5.1 as well as the decision why the specific type of the Euler angles, as the parameters recording material orientation to the curve $\mathbf{r}(S)$, were chosen.

For geometrical considerations the angular velocity of the coordinate system $C\xi\eta\zeta$ during the motion of Q from A to B, when the center C moves with unite velocity, is of considerable interest, see Biezeno and Grammel (1953). That is $\boldsymbol{\omega} = \dot{\boldsymbol{\psi}} + \dot{\boldsymbol{\vartheta}} + \dot{\boldsymbol{\varphi}} = \omega_\xi \mathbf{a} + \omega_\eta \mathbf{b} + \omega_\zeta \mathbf{c}$, where dot denotes the derivative with respect to S, i.e., $(\dot{\circ}) = d(\circ)/dS$ and where $\omega_\xi = \omega_\xi(S)$, $\omega_\eta = \omega_\eta(S)$ and $\omega_\zeta = \omega_\zeta(S)$. Therefore, the equations (5.1.1) and (5.1.2) of Subsection 5.1 will be recalled in the sequel. In these equations the derivatives with respect to s are involved. The derivatives with respect to S are easily obtained by use of (1.3.1). The elastic deformation of the column can be described by the components of curvature ω_ξ, ω_η, and ω_ζ that represents the twist around the axis ζ.

The effects of shear stresses will be described by use of the two angles, say $\alpha = \alpha(S)$ and $\beta = \beta(S)$. Namely, the unit vector normal to Q, here denoted by $\mathbf{c}$, is transformed to the unit vector of the tangent to the rod axis, say $\mathbf{t}$, by use of two rotations about the principal axis of the cross-section Q, as shown in Subsection 5.1. From the direct notation of $\mathbf{t}$ with respect to $Oxyz$

$$\mathbf{t} = \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{(1+\varepsilon)\, dS} = \frac{\dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k}}{(1+\varepsilon)}, \tag{1.3.2}$$

and its indirect notation with respect to the same coordinate system, this time by use of (5.1.6) and (5.1.3), the following differential relations between the functions $x(S)$, $y(S)$,

$z(S)$ and $\psi(S)$, $\vartheta(S)$, $\varphi(S)$, $\alpha(S)$, and $\beta(S)$ are obtained

$$\begin{aligned}
\dot{x} &= (1+\varepsilon)\,[\cos\beta\sin\alpha\cos\psi\cos\varphi + \cos\beta\sin\alpha\sin\varphi\sin\vartheta\sin\psi + \\
&\quad \sin\beta\sin\varphi\cos\psi - \sin\beta\cos\varphi\sin\vartheta\sin\psi + \cos\alpha\cos\beta\cos\vartheta\sin\psi], \\
\dot{y} &= (1+\varepsilon)\,[\cos\beta\sin\alpha\sin\varphi\cos\vartheta - \sin\beta\cos\varphi\cos\vartheta - \cos\alpha\cos\beta\sin\vartheta], \qquad (1.3.3) \\
\dot{z} &= (1+\varepsilon)\,[-\cos\beta\sin\alpha\sin\psi\cos\varphi + \cos\beta\sin\alpha\sin\varphi\sin\vartheta\cos\psi - \\
&\quad \sin\beta\sin\varphi\sin\psi - \sin\beta\cos\varphi\sin\vartheta\cos\psi + \cos\alpha\cos\beta\cos\vartheta\cos\psi],
\end{aligned}$$

As in the case of the compressibility of the rod axis the shear measures will be introduced in Subsection 1.5. For $\varepsilon = 0$ and $\alpha = \beta = 0$ the equations (1.3.3) coincide with the equations valid for the Kirchhoff theory derived in Spasic (1991, p.12).

We close this subsection by remark that the categories of the previous subsection representing the rod states could be described by the following relations: $x(S) = y(S) = 0$, $z(S) = S$, $\psi(S) = \vartheta(S) = \varphi(S) = 0$, $\alpha(S) = \beta(S) = 0$, $\omega_\xi(S) = \omega_\eta(S) = \omega_\zeta(S) = 0$ for the undeformed state; $x(S) = y(S) = 0$, $z(S) = s \neq S$, $\psi(S) = \vartheta(S) = 0$, $\varphi(S) \neq 0$, $\alpha(S) = \beta(S) = 0$, $\omega_\xi(S) = \omega_\eta(S) = 0$, $\omega_\zeta(S) \neq 0$ for the trivial configuration and finally $x(S) \neq 0$, $y(S) \neq 0$, $z(S) \neq s$, $\psi(S) \neq 0$, $\vartheta(S) \neq 0$, $\varphi(S) \neq 0$, $\alpha(S) \neq 0$, $\beta(S) \neq 0$, $\omega_\xi(S) \neq 0$, $\omega_\eta(S) \neq 0$, $\omega_\zeta(S) \neq 0$ for the nontrivial configuration.

2.4 Static equations

Consider an elementary part of the rod, of length ds, in equilibrium in the deformed state, at position s measured from A, shown in Fig. 1. Let the distributed force and the distributed torques per unit length in the deformed state be $\mathbf{q}$ and $\mathbf{m}$ respectively. The inner elastic forces in the cross-section denoted by s are reduced to the resultant force, say $\mathbf{V}$, and the resultant couple, say $\mathbf{M}$. Indices added to $\mathbf{V}$ and $\mathbf{M}$ denote the influence of the rejected parts of the rod. The usual conventions on the sign and differential of those quantities are assumed.

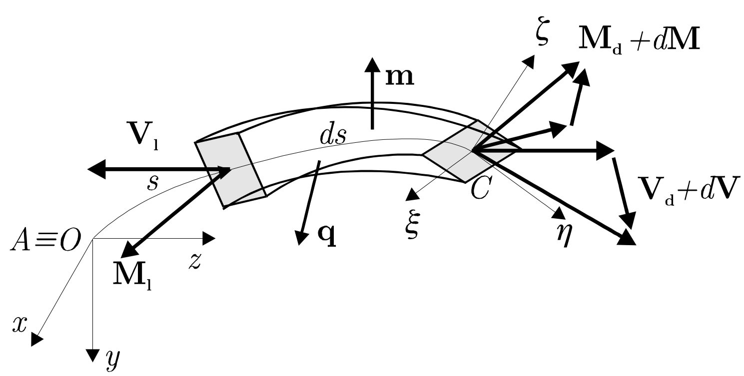


Fig. 1. Free body diagram for the rod element.

Static equations applied to the rod element state that sum of all forces and sum of all moments about the point with coordinate s vanish, i.e., $\mathbf{V}_l + \mathbf{V}_d + d\mathbf{V} + \mathbf{q}ds = 0$;

$\mathbf{M}_l + \mathbf{M}_d + d\mathbf{M} + \mathbf{m}ds + ds\mathbf{t} \times (\mathbf{V}_d + d\mathbf{V}) + \mathbf{t}ds \times \mathbf{q}ds/2 = 0$. Taking into account the sign convention, $\mathbf{V}_d = -\mathbf{V}_l = \mathbf{V}$ and $\mathbf{M}_d = -\mathbf{M}_l = \mathbf{M}$ and neglecting the second order terms yields $d\mathbf{V}/ds + \mathbf{q} = \mathbf{0}$; $d\mathbf{M}/ds + \mathbf{m} + \mathbf{t} \times \mathbf{V} = \mathbf{0}$. In this work it is assumed that there are neither static forces nor moments distributed along the column element ($\mathbf{q} = \mathbf{0},\ \mathbf{m} = \mathbf{0}$) so the static equations become $d\mathbf{V}/ds = 0$ and $d\mathbf{M}/ds + \mathbf{t} \times \mathbf{V} = \mathbf{0}$. These equations could be interpreted in either fixed $Oxyz$ or moving $C\xi\eta\zeta$ coordinate system. In the first case direct integration yields

$$\begin{gathered}
V_x = V_x(S) = const_1.\ \ M_x = M_x(S) = M_x(0) + V_y[z(S) - z(0)] - V_z[y(S) - y(0)], \\
V_y = V_y(S) = const_2.\ \ M_y = M_y(S) = M_y(0) + V_z[x(S) - x(0)] - V_x[z(S) - z(0)], \\
V_z = V_z(S) = const_3.\ \ M_z = M_z(S) = M_z(0) + V_x[y(S) - y(0)] - V_y[x(S) - x(0)],
\end{gathered} \tag{1.4.1}$$

where we used (1.3.2). In the second case, by use of (5.1.4) and (5.1.6), the static equations remain in the following differential form

$$\begin{gathered}
\dot{V}_\xi - V_\eta\omega_\zeta + V_\zeta\omega_\eta = 0, \\
\dot{V}_\eta - V_\zeta\omega_\xi + V_\xi\omega_\zeta = 0, \\
\dot{V}_\zeta - V_\xi\omega_\eta + V_\eta\omega_\xi = 0, \\
\dot{M}_\xi - M_\eta\omega_\zeta + M_\zeta\omega_\eta - [(1+\varepsilon)\cos\alpha\cos\beta]V_\eta - [(1+\varepsilon)\sin\beta]V_\zeta = 0, \\
\dot{M}_\eta - M_\zeta\omega_\xi + M_\xi\omega_\zeta + [(1+\varepsilon)\cos\alpha\cos\beta]V_\xi - [(1+\varepsilon)\sin\alpha\cos\beta]V_\zeta = 0, \\
\dot{M}_\zeta - M_\xi\omega_\eta + M_\eta\omega_\xi + [(1+\varepsilon)\sin\alpha\cos\beta]V_\eta + [(1+\varepsilon)\sin\beta]V_\xi = 0,
\end{gathered} \tag{1.4.2}$$

where we used usual notation for the projections of the vector on axes. The above relations are derived without any constrains on geometrical variables and are generalizations of the plane elastica relations given in Reissner (1972, p. 797). In the classical case when $\varepsilon = 0$, $\alpha = \beta = 0$, the equations (1.4.2) coincide with the equations originally derived by Kirchhoff, see Filin *et al.* (1983, p.31). For the small values of the shear angles α and β, (when $\sin\alpha \approx \alpha$, $\sin\beta \approx \beta$, $\cos\alpha, \cos\beta \approx 1$) the equations given in Dupuis & Rousselet (1992, p. 478) where the problem of motion of curved pipes conveying fluid are obtained as the special case of (1.4.2).

REMARK: In relations (1.4.2) dot still denotes the derivative with respect to S, since according to (1.3.1) in the static equations s was replaced by S. The components ω_ξ, ω_η and ω_ζ in (1.4.2) are of the form (5.1.1) where derivatives with respect to S are used. Namely, the relations (5.1.1) and (5.1.4) were written by use of derivatives with respect to s and then transformed to the form which contain the derivatives with respect to S, where the term $(1+\varepsilon)$ was eliminated and left only in the second term of the moment equation.

2.5 Constitutive equations

In order the fifteen unknown functions (introduced in Subsections 1.2 and 1.3) to be determined, it is necessary to consider six more equations besides (1.3.3) and either (1.4.1) or (1.4.2). New equations should connect the external load with parameters describing the spatial deformation of the rod and will be introduced, on the basis of the suggested physical model, as proposed by Eliseyev (1998). Namely, as a measure deformation two vectors are introduced: the vector $\mathbf{T} = d\mathbf{r}/dS - \mathbb{P}\mathbf{k} = d\mathbf{r}/dS - \mathbf{c}$, where $\mathbb{P}$ denotes the deformation tensor of the moving system, defined by (5.1.5), and the vector of the angular velocity $\mathbf{W}$, representing absolute rotation of the cross-section Q, defined by $\mathbf{W} = \omega_\xi \mathbf{a} + \omega_\eta \mathbf{b} + \omega_\zeta \mathbf{c}$, where ω_ξ, ω_η and ω_ζ are given by (5.1.1) in which derivatives with respect to S are taken. The projections of $\mathbf{T}$ on axes $C\xi$ and $C\eta$ are measures of the shear deformation and the projection of $\mathbf{T}$ on $C\zeta$ axis is the measure of compressibility.

The resultant force and the resultant couple in an arbitrary section of the rod are postulated in the moving coordinate system in the following way

$$\mathbf{M} = \mathbf{AW}, \qquad \mathbf{V} = \mathbf{BT}, \tag{1.5.1}$$

where $\mathbf{A}$ and $\mathbf{B}$ denote symmetrical stiffness tensors which connect elastic properties of the material with the shape of the cross section. These equations correspond to the rod which is straight and prismatic in an undeformed (unloaded, initial) state. If the elements of the tensors $\mathbf{A}$ and $\mathbf{B}$ are chosen to be constant, then in sense of the generalized Bernoulli-Euler elastica, on the basis of Love (1927) and several research results of the generalized plane elastica theory shown in (1.1.1-4), the constitutive equations for spatially deformed rod in this notation read

$$\begin{aligned}
V_\xi &= kGA\,(1+\varepsilon)\cos\beta\sin\alpha, & M_\xi &= EI\omega_\xi, \\
V_\eta &= -hGA\,(1+\varepsilon)\sin\beta, & M_\eta &= EJ\omega_\eta, \\
V_\zeta &= EA\left[(1+\varepsilon)\cos\alpha\cos\beta - 1\right], & M_\zeta &= GJ\omega_\zeta.
\end{aligned} \tag{1.5.2}$$

In the above relations GA and EA represent shear and extensional rigidity. The coefficients k and h are the shear correction factors depending on the shape of the cross-section and the material of the rod, see Renton (1991). The quantities EI, EJ and GJ are bending and torsional rigidities for principal axis of the cross section Q, respectively. In engineering applications it is often assumed that EA and GA are modulus of elasticity and shear modulus multiplied by the area of the cross-section respectively, EI and EJ are modulus of elasticity multiplied by the moments of inertia for principal axes of inertia respectively and that GJ is shear modulus multiplied by the moment of inertia under torsion (for circular cross-section the moment of inertia about polar axis). Thus, for circular, quadratic and rectangular (with relatively small ratio of the height and width), cross-section and for materials with the Poisson ratio 0.3 it is worth noting that $EI \geq EJ > GJ$, while for rectangular cross-section with higher ratio of the height and width reads $EI > EJ < GJ$.

For the case when $\varepsilon = 0$, $\alpha = \beta = 0$, what corresponds to infinite values of extensional and shear rigidity or to the case $\mathbf{T} = \mathbf{0}$, the equations (1.5.2), coincide with the well known classical equations first derived by Kirchhoff (see Filin *et al.* (1983, p. 37). The linearized version of (1.5.2), for small angles α and β , i.e., $\sin\alpha \approx \alpha$, $\sin\beta \approx \beta$, $\cos\alpha \approx \cos\beta \approx 1$, and $\varepsilon\alpha \approx \varepsilon\beta \approx 0$, is used in Dupius & Rousselet (1992, p. 479). In the case of planar deformation of the rod ($\beta = \psi = \varphi = 0$, $\alpha \equiv \gamma$) the equations (1.5.2) coincide with the equations (1.1.2) presented in Goto *et al.* (1990) and represent their direct generalization. According to the classification mentioned in Subsection 1.1 the suggested rod model (1.5.2) fulfills into Haringx's approach of the decomposition. In the general context of elastic rods, see Antman (1972), we may say that the Eliseyev model corresponds to the Cosserat model with orthogonal basis directions.

2.6 Boundary conditions

According to the assumed supports the reactions in A and B are introduced as $\mathbf{V}_A$ and $\mathbf{V}_B$ ($\mathbf{M}_A$, $\mathbf{M}_B = 0$), and the static equations of the rod are written: $\sum X_i = V_{Ax} + V_{Bx} = 0$, $\sum Y_i = V_{Ay} + V_{By} = 0$, $\sum Z_i = V_{Az} - P = 0$, $\sum M_x = -V_{By}\overline{AB} = 0$, $\sum M_y = V_{Bx}\overline{AB} = 0$, $\sum M_z = W - W = 0$. These equations with the sign convention ($V_z(0) = -V_{Az}$, $V_x(L) = V_{Bx}$, $V_y(L) = V_{By}$, $M_z(0) = M_z(L) = W$, $M_x(L) = M_{Bx}$, $M_y(L) = M_{By}$) yield the following physical conditions

$$
\begin{gathered}
V_x(L) = 0, \quad M_x(0) = 0, \\
V_y(L) = 0, \quad M_y(0) = 0, \\
V_z(0) = -P, \; M_z(0) = W.
\end{gathered}
\tag{1.6.1}
$$

The corresponding geometrical conditions are

$$
\begin{gathered}
x(0) = 0, \; x(L) = 0, \\
y(0) = 0, \; y(L) = 0, \\
z(0) = 0, \; \varphi(L) = 0,
\end{gathered}
\tag{1.6.2}
$$

see Grammel (1923).

2.7 Nonlinear differential equations describing equilibrium of compressed and twisted linearly elastic rod

In this subsection the system of nonlinear ordinary differential equations describing the equilibrium of spatially deformed compressed and twisted rod is obtained from the equations presented in Subsections 1.2-6 and 5.1. The attribute nonlinear follows from geometrical considerations, see Panovko & Gubanova (1987, p. 10). In order to write that system in dimensionless form the following dimensionless quantities are introduced:

$$
\begin{gathered}
S = \frac{S}{L}, \qquad x = \frac{x}{L}, \qquad y = \frac{y}{L}, \qquad z = \frac{z}{L}, \\
\omega_\sigma = \omega_\sigma L, \qquad M_\sigma = \frac{M_\sigma}{PL}, \qquad V_\sigma = \frac{V_\sigma}{P}, \qquad (\sigma = x, y, z \quad \text{or} \quad \xi, \eta, \zeta) \\
\lambda = \frac{W}{PL}, \qquad e = \frac{kGA}{P}, \qquad f = \frac{hGA}{P}, \qquad g = \frac{EA}{P}, \\
a = \frac{EI}{PL^2}, \qquad b = \frac{EJ}{PL^2}, \qquad c = \frac{GJ}{PL^2},
\end{gathered}
\tag{1.7.1}
$$

where the same notation is used for dimensionless S, x, y, z, $\mathbf{V}$ and $\mathbf{M}$. By use of (1.7.1) the equations of previous subsections will be included in the following directly i.e., without noting their dimensionless form.

The main subject of this work is the analysis of the compressed and twisted rod so from now on it will be assumed that $P \neq 0$. In Spasic (1991) the case $P = 0$ was considered together with the corresponding form of dimensionless quantities.

Introducing (1.7.1) into (1.6.1), (1.6.2) and substituting into (1.4.1) for each $S \in [0, 1]$ we get

$$
\begin{gathered}
V_x = 0, \quad M_x = y, \\
V_y = 0, \quad M_y = -x, \\
V_z = -1, \quad M_z = \lambda.
\end{gathered}
\tag{1.7.2}
$$

On the basis of these equations, if $x = x\,(S)$ and $y = y\,(S)$ are known, the load of the rod along its axis will be completely determined. In order to find these functions, as well as the remaining functions that describe the position and orientation of the spatially buckled rod, (5.1.3) and (1.7.2) will be connected first. Namely, this leads to the following relations that are valid for each $S \in [0,1]$

$$V_\xi = -\left(-\sin\psi\cos\varphi + \sin\varphi\sin\vartheta\cos\psi\right),$$

$$V_\eta = -\left(\sin\varphi\sin\psi + \cos\varphi\sin\vartheta\cos\psi\right),$$

$$V_\zeta = -\cos\vartheta\cos\psi,$$

$$\begin{gathered} M_\xi = y\left(\cos\psi\cos\varphi + \sin\varphi\sin\vartheta\sin\psi\right) - x\sin\varphi\cos\vartheta + \\ \lambda\left(-\sin\psi\cos\varphi + \sin\varphi\sin\vartheta\cos\psi\right), \end{gathered} \tag{1.7.3}$$

$$\begin{gathered} M_\eta = y\left(-\sin\varphi\cos\psi + \cos\varphi\sin\vartheta\sin\psi\right) - x\cos\varphi\cos\vartheta + \\ \lambda\left(\sin\varphi\sin\psi + \cos\varphi\sin\vartheta\cos\psi\right), \end{gathered}$$

$$M_\zeta = y\cos\vartheta\sin\psi + x\sin\vartheta + \lambda\cos\vartheta\cos\psi.$$

Introducing the constitutive equations (1.5.2) into (1.7.3) and (1.3.3), eliminating α, β and ε, and using (5.1.2), after some elementary calculations (all of them done in the dimensionless form), the equilibrium equations in the fixed system $Oxyz$ are derived in the following form

$$
\begin{aligned}
\dot{x} = &\left(\cos\psi\cos\varphi+\sin\varphi\sin\vartheta\sin\psi\right)\left(\sin\psi\cos\varphi-\sin\varphi\sin\vartheta\cos\psi\right)/e+\\
&\left(\sin\varphi\sin\psi+\cos\varphi\sin\vartheta\cos\psi\right)\left(\sin\varphi\cos\psi-\cos\varphi\sin\vartheta\sin\psi\right)/f+\\
&\left(1-\cos\vartheta\cos\psi/g\right)\cos\vartheta\sin\psi,\\[1em]
\dot{y} = &\left(\sin\varphi\cos\vartheta\right)\left(\sin\psi\cos\varphi-\sin\varphi\sin\vartheta\cos\psi\right)/e-\\
&\left(\sin\varphi\sin\psi+\cos\varphi\sin\vartheta\cos\psi\right)\left(\cos\varphi\cos\vartheta\right)/f-,\\
&\left(1-\cos\vartheta\cos\psi/g\right)\sin\vartheta,\\[1em]
\dot{z} = &\left(-\sin\psi\cos\varphi+\sin\varphi\sin\vartheta\cos\psi\right)\left(\sin\psi\cos\varphi-\sin\varphi\sin\vartheta\cos\psi\right)/e-\\
&\left(\sin\varphi\sin\psi+\cos\varphi\sin\vartheta\cos\psi\right)\left(\sin\varphi\sin\psi+\cos\varphi\sin\vartheta\cos\psi\right)/f+\\
&\left(1-\cos\vartheta\cos\psi/g\right)\cos\vartheta\cos\psi,\\[1em]
\dot{\psi} = &\frac{1}{\cos\vartheta}\left\{\left(\frac{1}{a}-\frac{1}{b}\right)\sin\varphi\cos\varphi\left(y\cos\psi-\lambda\sin\psi\right)+\right.\\
&\left.\left(\frac{\sin^2\varphi}{a}+\frac{\cos^2\varphi}{b}\right)\left[\left(y\sin\psi+\lambda\cos\psi\right)\sin\vartheta-x\cos\vartheta\right]\right\},\\[1em]
\dot{\vartheta} = &\left(\frac{\cos^2\varphi}{a}+\frac{\sin^2\varphi}{b}\right)\left(y\cos\psi-\lambda\sin\psi\right)+\\
&\left(\frac{1}{a}-\frac{1}{b}\right)\sin\varphi\cos\varphi\left[\sin\vartheta\left(y\sin\psi+\lambda\cos\psi\right)-x\cos\vartheta\right]\\[1em]
\dot{\varphi} = &\cos\vartheta\left(y\sin\psi+\lambda\cos\psi\right)/c+\left[\frac{x}{c}+\left[\dot{\psi}\right]\right]\sin\vartheta,\\[1em]
&x(0)=0,\qquad y(0)=0,\qquad z(0)=0,\qquad \varphi(0)=0,\\[1em]
&x(1)=0,\qquad y(1)=0,
\end{aligned}
\tag{1.7.4}
$$

where dot denotes the derivative with respect to the dimensionless arc length S and where the boundary conditions (1.6.2) were applied.

The solution of the boundary value problem (1.7.4) completely determines the geometry of the rod in the deformed state, as well as the quantities describing shear and compressibility. Namely, to find ε, α and β it is necessary to connect $(1.7.3)_{3,4,5}$ with

$(1.5.2)_{1,2,3}$, as a result one obtains

$$\tan\alpha = \frac{(\sin\psi\cos\varphi - \sin\varphi\sin\vartheta\cos\psi)\, g}{e\,(g - \cos\vartheta\cos\psi)},$$

$$\tan\beta = \frac{(\sin\varphi\sin\psi + \cos\varphi\sin\vartheta\cos\psi)\, g}{f\,(g - \cos\vartheta\cos\psi)}\,[\cos\alpha]\,, \tag{1.7.5}$$

$$\varepsilon = \frac{g - \cos\vartheta\cos\psi}{g\,[\cos\alpha]\,[\cos\beta]} - 1.$$

For all values $\lambda > 0$ the nonlinear system (1.7.4) admits a trivial solution that corresponds to the trivial equilibrium configuration: $x(S) = y(S) = 0$, $\psi(S) = \vartheta(S) = 0$, $z(S) = (1 - 1/g)\,S$,and $\varphi(S) = \lambda S/c$. We intend to determine the smallest value of $\lambda = \lambda_{cr}$ for which (1.7.4) has more than one solution. Namely, the couple W will be treated as the bifurcation parameter: that is the quantities P, L, k, h, EA, GA, EI, EJ, GJ will be fixed. Physically this corresponds to the situation when the rod is compressed with the known force P and the torsional couple W is varied until bending. From the point of view of the qualitative analysis in such a way we avoid the technical difficulty of having the bifurcation parameter set to be in $\mathbb{R}^2$ (which would happen if the physically natural choice of P and W as the bifurcation parameters was followed).

According to the Euler criterion the value $\lambda = \lambda_{cr}$ i.e., $W = W_{cr}$, for which (1.7.4) has a nontrivial solution determines the stability boundary. Thus, determining the bifurcation points of the nonlinear system (1.7.4) occupies the central position in the stability analysis that follows. The system (1.7.4) is of six order, with right hand sides continuous and bounded on the interval $[0, 1]$. For the system of second order and the same conditions imposed on the right hand sides, a very interesting analysis of the bifurcation points based on the geometrical argument is given Krasnoselskii *et al.* (1963, p. 204). That argument states that the stable eigenvalues of the linearized problem determine the eigenvalues of the nonlinear problem. To recognize the stable eigenvalue we use the solution of the linearized problem to define the angle function and ask the condition that the eigenvalue under consideration does not make the angle function at right boundary to has its local extrema, see Krasnoselskii *et al.* (1963, p.178). Despite the fact that line of argument is easy to apply to stability problems for elastic rods, see Spasic and Glavardanov (1993), the generalization needed for the system (1.7.4) is rather difficult. Thus, the bifurcation of the nonlinear equilibrium equations will be performed on a more simple but equivalent system.

Substituting (1.7.1) and (1.5.2) into the static equations presented in the moving coordinate system $C\xi\eta\zeta$, (1.4.2), yields the following system of nonlinear differential equations describing the equilibrium configuration of the compressed and twisted linearly-elastic rod

$$
\begin{gathered}
\dot{V}_\xi - V_\eta \omega_\zeta + V_\zeta \omega_\eta = 0, \\
\dot{V}_\eta - V_\zeta \omega_\xi + V_\xi \omega_\zeta = 0, \\
\dot{V}_\zeta - V_\xi \omega_\eta + V_\eta \omega_\xi = 0, \\
a\dot{\omega}_\xi - b\omega_\eta \omega_\zeta + c\omega_\zeta \omega_\eta - \left(1 + \frac{V_\zeta}{g}\right) V_\eta - \frac{V_\eta V_\zeta}{f} = 0, \\
b\dot{\omega}_\eta - c\omega_\zeta \omega_\xi + a\omega_\xi \omega_\zeta + \left(1 + \frac{V_\zeta}{g}\right) V_\xi - \frac{V_\xi V_\zeta}{e} = 0, \\
c\dot{\omega}_\zeta - a\omega_\xi \omega_\eta + b\omega_\eta \omega_\xi + V_\xi V_\eta \left(\frac{1}{e} - \frac{1}{f}\right) = 0.
\end{gathered} \tag{1.7.6}
$$

In order to get the corresponding boundary conditions, on the basis of $(1.5.2)_{4,5,6}$ and (1.6.2), (1.7.3) will be estimated at $S = 0$ and $S = 1$, see Spasic (1991, p. 18), that is

$$
\begin{array}{ll}
\lambda V_\xi (0) + a\omega_\xi (0) = 0, & \lambda V_\xi (1) + a\omega_\xi (1) = 0, \\
\lambda V_\eta (0) + a\omega_\eta (0) = 0, & \lambda V_\eta (1) + a\omega_\eta (1) = 0, \\
\lambda V_\zeta (0) + a\omega_\zeta (0) = 0, & \lambda V_\zeta (1) + a\omega_\zeta (1) = 0.
\end{array} \tag{1.7.7}
$$

The trivial solution of the boundary value problem (1.7.6-7): $V_\xi (S) = V_\eta (S) = 0$, $V_\zeta (S) = -1$, $\omega_\xi (S) = \omega_\eta (S) = 0$, $\omega_\zeta (S) = \lambda / c$, corresponds to the trivial equilibrium configuration. The next task is to examine the existence and uniqueness of nontrivial solutions to that problem. That will be done in the next subsection. Before we proceed to it we made two preparatory results.

First, on the basis of purely geometrical argument and (1.7.2) we get the following first integrals

$$
\begin{gathered}
V_\xi^2 + V_\eta^2 + V_\zeta^2 = 1 = \mathbf{V} \cdot \mathbf{V} = V_x^2 + V_y^2 + V_z^2, \\
a\omega_\xi V_\xi + b\omega_\eta V_\eta + c\omega_\zeta V_\zeta = -\lambda = \mathbf{M} \cdot \mathbf{V} = M_x V_x + M_y V_y + M_z V_z,
\end{gathered} \tag{1.7.8}
$$

which reduces the order of the considered system. In (1.7.8) the usual notation for scalar product is used. Note, that as expected for the infinite values of the extensional and shear rigidities (e, f, $g \to \infty$) the system (1.7.4) and (1.7.6) reduces to the form generated by the classical Kirchhoff theory, see Grammel (1923) and Spasic (1991, p. 17) respectively. In that special case in the moving system there exists one more first ("energy") integral

$$
a\omega_\xi^2 + b\omega_\eta^2 + c\omega_\zeta^2 + 2V_\zeta = const. = B, \tag{1.7.9}
$$

which could be used, for the further reduction of the system order, as well as for the control of the numerical procedures connected with the solution of (1.7.6), see Vujanovic (1992, p. 322).

Secondly, by use of nonsingular transformations the new variables

$$\begin{gathered} u_1 = V_\xi, \qquad u_2 = V_\eta, \qquad u_3 = V_\zeta + 1, \\ u_4 = \lambda V_\xi + a\omega_\xi, \qquad u_5 = \lambda V_\eta + a\omega_\eta, \qquad u_6 = \lambda V_\zeta + a\omega_\zeta, \end{gathered} \tag{1.7.10}$$

will be introduced. These variables measure the displacements with respect to the trivial configuration, (all of them vanish on the trivial configuration). Next, if we assume that $\psi,\ \vartheta \in [-\pi/2,\ \pi/2]$ then from (1.7.8) one finds

$$\begin{aligned} V_\zeta &= -\left(1 - V_\xi^2 - V_\eta^2\right)^{1/2}, \\ \omega_\zeta &= \frac{\lambda + a\omega_\xi V_\xi + b\omega_\eta V_\eta}{c\left(1 - V_\xi^2 - V_\eta^2\right)^{1/2}}, \end{aligned} \tag{1.7.11}$$

or

$$u_3 = -\left(1 - u_1^2 - u_2^2\right)^{1/2},$$

$$u_6 = (u_1 u_4 + u_2 u_5)\left(1 - u_1^2 - u_2^2\right)^{-1/2}.$$

With the new variables and the first integrals involved, the boundary value problem

(1.7.6-7) becomes

$$\begin{aligned}
\dot{u}_1 =& \lambda\left(\frac{1}{c}-\frac{1}{b}\right)u_2+\frac{u_5}{b}- \\
& \left[\lambda\left(\frac{1}{c}-\frac{1}{b}\right)u_2+\frac{u_5}{b}\right]\left[1-\left(1-u_1^2-u_2^2\right)^{1/2}\right]+ \\
& \frac{u_2\left(u_1u_4+u_2u_5\right)\left(1-u_1^2-u_2^2\right)^{-1/2}}{c}, \\
\dot{u}_2 =& \lambda\left(\frac{1}{a}-\frac{1}{c}\right)u_1-\frac{u_4}{a}+ \\
& \left[\lambda\left(\frac{1}{c}-\frac{1}{a}\right)u_1+\frac{u_4}{a}\right]\left[1-\left(1-u_1^2-u_2^2\right)^{1/2}\right]+ \\
& \frac{u_1\left(u_1u_4+u_2u_5\right)\left(1-u_1^2-u_2^2\right)^{-1/2}}{c}, \\
\dot{u}_4 =& \left(1+\frac{1}{f}-\frac{1}{g}\right)u_2+\lambda\frac{u_5}{c}- \\
& \left[\left(\frac{1}{f}-\frac{1}{g}\right)u_2+\lambda\frac{u_5}{c}\right]\left[1-\left(1-u_1^2-u_2^2\right)^{1/2}\right]+ \\
& \left[\left(\frac{1}{c}-\frac{1}{b}\right)u_5+\lambda\frac{u_2}{b}\right]\left(u_1u_4+u_2u_5\right)\left(1-u_1^2-u_2^2\right)^{-1/2}, \\
\dot{u}_5 =& -\left(1+\frac{1}{e}-\frac{1}{g}\right)u_1-\lambda\frac{u_4}{c}+ \\
& \left[\left(\frac{1}{e}-\frac{1}{g}\right)u_1+\lambda\frac{u_4}{c}\right]\left[1-\left(1-u_1^2-u_2^2\right)^{1/2}\right]- \\
& \left[\left(\frac{1}{c}-\frac{1}{a}\right)u_4+\lambda\frac{u_1}{a}\right]\left(u_1u_4+u_2u_5\right)\left(1-u_1^2-u_2^2\right)^{-1/2},
\end{aligned} \tag{1.7.12}$$

with

$$\begin{aligned}
u_4(0) = u_4(1) = 0, \\
u_5(0) = u_5(1) = 0.
\end{aligned} \tag{1.7.13}$$

With this preparation the qualitative analysis of the problem (1.7.12-13) will be done in the next subsection.

2.8 Linear vs. nonlinear problem

This subsection is concerned with the operator form of the equilibrium equations, the eigenvalues of the linearized equations and the bifurcation theorem for the nonlinear system.

We intend to analyze does the system of nonlinear equilibrium equations describing the spatially deformed compressed and twisted column have nontrivial solutions. First, the corresponding nonlinear boundary value problem will be written in the compact (operator) form. Then, the eigenfunctions of the linearized problem will be examined. The theory of bifurcation based on linearized equations will be used to establish the existence of nontrivial equilibrium configurations in a neighbourhood of the trivial one. In doing so the methods of nonlinear analysis presented in Chow & Hale (1982, Ch. V), especially the Liapunov-Schmidt procedure, will be followed. In order to apply these methods further preparation of the rod problem is needed. Thus, the line of argument shown in Atanackovic (1989), where the buckling problem of the compressible column is treated, will be useful.

Consider the Sobolev space

$$H^1 = \left\{ u \mid u = (u_1, u_2, u_4, u_5)^{\mathrm{T}}; \quad \int_0^1 u u^{\mathrm{T}} dS < \infty; \quad \int_0^1 \dot{u}\dot{u}^{\mathrm{T}} dS < \infty \right\},$$

consisting of those functions $u : [0,1] \to \mathbb{R}^4$ that are square integrable and have a square integrable generalized derivative. Further, let H_0^1 be the subspace of H^1 defined by

$$H_0^1 = \left\{ u \mid u \in H^1; \quad u_4(0) = u_5(0) = u_4(1) = u_5(1) = 0 \right\}.$$

Also let the Q be the space of the square integrable functions $q : [0,1] \to \mathbb{R}^4$ i.e.,

$$Q = \left\{ q \mid q = (q_1, q_2, q_4, q_5)^{\mathrm{T}}; \quad \int_0^1 q q^{\mathrm{T}} dS < \infty \right\}.$$

Let $\mathcal{F}$ be the non-linear operator depending on the parameter $\lambda \geq 0$ with the domain $\mathbb{R} \times H_0^1$ and the range Q defined as

$$\mathcal{F}(\lambda, u) = \dot{u} + \mathcal{K}(\lambda) u + \mathcal{N}(\lambda, u), \tag{1.8.1}$$

where

$$\mathcal{K}(\lambda) = \begin{bmatrix} 0 & \lambda\left(\dfrac{1}{b} - \dfrac{1}{c}\right) & 0 & -\dfrac{1}{b} \\ \lambda\left(\dfrac{1}{c} - \dfrac{1}{a}\right) & 0 & \dfrac{1}{a} & 0 \\ 0 & \left(\dfrac{1}{g} - \dfrac{1}{f} - 1\right) & 0 & -\dfrac{\lambda}{c} \\ \left(1 + \dfrac{1}{e} - \dfrac{1}{g}\right) & 0 & \dfrac{\lambda}{c} & 0 \end{bmatrix}, \qquad \text{and}$$

$$\mathcal{N}(\lambda,u)=\begin{bmatrix}\dfrac{\left[\lambda\left(\frac{1}{c}-\frac{1}{b}\right)u_2+\frac{u_5}{b}\right]\left[1-\left(1-u_1^2-u_2^2\right)^{1/2}\right]-u_2\left(u_1u_4+u_2u_5\right)\left(1-u_1^2-u_2^2\right)^{-1/2}}{c}\\[2ex] \dfrac{-\left[\lambda\left(\frac{1}{c}-\frac{1}{a}\right)u_1+\frac{u_4}{a}\right]\left[1-\left(1-u_1^2-u_2^2\right)^{1/2}\right]+u_1\left(u_1u_4+u_2u_5\right)\left(1-u_1^2-u_2^2\right)^{-1/2}}{c}\\[2ex] \left[\left(\frac{1}{f}-\frac{1}{g}\right)u_2+\lambda\frac{u_5}{c}\right]\left[1-\left(1-u_1^2-u_2^2\right)^{1/2}\right]-\left[\left(\frac{1}{c}-\frac{1}{b}\right)u_5+\lambda\frac{u_2}{b}\right]\left(u_1u_4+u_2u_5\right)\left(1-u_1^2-u_2^2\right)^{-1/2}\\[2ex] -\left[\left(\frac{1}{e}-\frac{1}{g}\right)u_1+\lambda\frac{u_4}{c}\right]\left[1-\left(1-u_1^2-u_2^2\right)^{1/2}\right]+\left[\left(\frac{1}{c}-\frac{1}{a}\right)u_4+\lambda\frac{u_1}{a}\right]\left(u_1u_4+u_2u_5\right)\left(1-u_1^2-u_2^2\right)^{-1/2}\end{bmatrix}.$$

It is obvious that $\mathcal{N}(\lambda,u)=-\mathcal{N}(\lambda,-u)$, what as a consequence yields

$$\mathcal{F}(\lambda,u)=-\mathcal{F}(\lambda,-u). \tag{1.8.2}$$

The two point boundary value problem (1.7.12-13) now becomes

$$\mathcal{F}(\lambda,u)=0. \tag{1.8.3}$$

The point $u=0$ is the trivial solution of (1.8.3) for all values of λ. Following the idea of the Euler criterion for stability analysis it will be of interest to examine the solutions $u=\bar{u}$ that are in a neighbourhood of the solution $u=0$ for the values $\lambda>0$. If such solutions exist the trivial configuration is not stable because the equilibrium could be attained in two different positions. The existence result will be obtained as a direct application of the bifurcation theory developed on the linearized equations. Hence, let

$$\mathcal{L}(\lambda)=\mathcal{DF}(\lambda,0),$$

denotes the Fréchet derivative of the operator $\mathcal{F}$ at the point $u=0$. It is obvious that

$$\mathcal{L}(\lambda)=\dot{u}+\mathcal{K}(\lambda)u, \tag{1.8.4}$$

with $\mathcal{K}(\lambda)$ as above. If we introduce the following norms on H_0^1 and Q

$$\|u\|_1=\left[\int_0^1\left(\dot{u}\dot{u}^{\mathrm{T}}+uu^{\mathrm{T}}\right)dS\right]^{1/2},\qquad \|q\|_2=\left[\int_0^1\left(qq^{\mathrm{T}}\right)dS\right]^{1/2},$$

then $\mathcal{L}(\lambda)$ is a bounded linear operator with domain $\mathbb{R}\times H_0^1$ and the range Q.

We consider now the eigenvalues and the eigenfunctions of the linearized boundary value problem

$$\mathcal{L}(\lambda)\, u = 0, \tag{1.8.5}$$

which in explicit form reads

$$\begin{aligned}
\dot{u}_1 &= \lambda\left(\frac{1}{c}-\frac{1}{b}\right)u_2+\frac{u_5}{b},\\
\dot{u}_2 &= \lambda\left(\frac{1}{a}-\frac{1}{c}\right)u_1-\frac{u_4}{a},\\
\dot{u}_4 &= \left(1+\frac{1}{f}-\frac{1}{g}\right)u_2+\lambda\frac{u_5}{c}, \qquad u_4(0)=u_4(1)=0,\\
\dot{u}_5 &= -\left(1+\frac{1}{e}-\frac{1}{g}\right)u_1-\lambda\frac{u_4}{c}, \qquad u_5(0)=u_5(1)=0.
\end{aligned} \tag{1.8.6}$$

The nontrivial solution of the system (1.8.6) is assumed in the form

$$\begin{aligned}
u_1 &= \sum_{i=0}^{\infty} u_i^1 S^i,\, u_2=\sum_{i=0}^{\infty} u_i^2 S^i,\\
u_4 &= \sum_{i=0}^{\infty} u_i^4 S^i,\, u_5=\sum_{i=0}^{\infty} u_i^5 S^i,
\end{aligned} \tag{1.8.7}$$

where the coefficients u_i^j, $j=1,2,4,5$, $i=0,1,2,...$, are expressed as

$$\begin{aligned}
u_i^1 &= \frac{1}{i}\left[\lambda\left(\frac{1}{c}-\frac{1}{b}\right)u_{i-1}^2+\frac{u_{i-1}^5}{b}\right],\\
u_i^2 &= \frac{1}{i}\left[\lambda\left(\frac{1}{a}-\frac{1}{c}\right)u_{i-1}^1-\frac{u_{i-1}^4}{a}\right],\\
u_i^4 &= \frac{1}{i}\left[\left(1+\frac{1}{f}-\frac{1}{g}\right)u_{i-1}^2+\lambda\frac{u_{i-1}^5}{c}\right],\\
u_i^5 &= \frac{1}{i}\left[\left(\frac{1}{g}-1-\frac{1}{e}\right)u_{i-1}^1-\lambda\frac{u_{i-1}^4}{c}\right],
\end{aligned} \tag{1.8.8}$$

with the upper index denoting the variables. Obviously $u_4(0)=u_0^4=0$, $u_5(0)=u_0^5=0$, further, it could be assumed that $u_0^1=1$ while $\lambda=\lambda_{cr}>0$ and the coefficient u_0^2 which ensures the existence of the nontrivial solution of (1.8.6) could be determined from the boundary conditions

$$u_4(1) = \sum_{i=1}^{\infty} u_i^4 = 0, \qquad u_5(1) = \sum_{i=1}^{\infty} u_i^5 = 0. \tag{1.8.9}$$

By numerical examination of the determined coefficients it could be shown, see Liashko *et al.* (1986, p. 80), that the radii of convergence of the series (1.8.7) are greater than 1. The critical load parameter $\lambda = \lambda_{cr}$ at which the buckling starts is the minimal value of λ that satisfies (1.8.9). The numerical examination of the derivatives of (1.8.9) with respect to λ estimated at $\lambda = \lambda_{cr}$ for the corresponding value of u_0^2 leads to the conclusion that λ_{cr} represents a simple zero of the linearized system. Also, λ_{cr} determines the coefficients (1.8.8) uniquely so the eigenvector of (1.8.6) could be written as

$$u_s = C \begin{bmatrix} \sum\limits_{i=0}^{\infty} u_i^1 S^i \\ \\ \sum\limits_{i=0}^{\infty} u_i^2 S^i \\ \\ \sum\limits_{i=0}^{\infty} u_i^4 S^i \\ \\ \sum\limits_{i=0}^{\infty} u_i^5 S^i \end{bmatrix}, \tag{1.8.10}$$

and could be made unique if the constant C is determined from the condition $\|u_s\|_1 = 1$.

REMARK: In Spasic (1991, p. 27) the linearized system that corresponds to (1.8.6), (which could be obtained from (1.8.6) for infinite values of shear and extensional rigidities) was solved by use of the classical procedure for finding eigenvectors of differential equations with constant coefficients in the closed form, see Filipov (1979, p.57). That line of argument, although more complicated but equivalent, could be followed here as well. Then, instead of solving (1.8.9) the critical load could be determined from two characteristic algebraic equations that follow from two different assumed solution forms of the corresponding linearized problem. The technical difficulties that follow the genesis of such equations as well as their complexity did determine the argument which was adopted here. Besides that solutions in form of series compared to solutions in their closed forms in sense of numerical analysis do have some advantages.

To characterize the range of $\mathcal{L}(\lambda_{cr})$ we consider the operator $\mathcal{L}^*(\lambda_{cr})$ defined by

$$\langle \mathcal{L}(\lambda_{cr})\, u, q \rangle = \langle u, \mathcal{L}^*(\lambda_{cr})\, q \rangle,$$

where $\langle u, q \rangle = \int_0^1 u q^{\mathrm{T}} dS$. An easy calculation using partial integration shows that the equation

$$\mathcal{L}^*(\lambda_{cr})\, q = 0,$$

is equivalent to

$$
\begin{aligned}
\dot{q}_1 &= -\lambda\left(\frac{1}{a}-\frac{1}{c}\right)q_2-\left(\frac{1}{g}-1-\frac{1}{e}\right)q_5, \qquad q_1(0)=q_1(1)=0,\\
\dot{q}_2 &= -\lambda\left(\frac{1}{c}-\frac{1}{b}\right)q_1-\left(1+\frac{1}{f}-\frac{1}{g}\right)q_4, \qquad q_2(0)=q_2(1)=0,\\
\dot{q}_4 &= \frac{q_2}{a}+\lambda\frac{q_5}{c},\\
\dot{q}_5 &= -\frac{q_1}{b}-\lambda\frac{q_4}{c}.
\end{aligned}
\tag{1.8.11}
$$

The nontrivial solution of the system (1.8.11) is assumed in the form

$$
\begin{aligned}
q_1 &= \sum_{i=0}^{\infty} q_i^1 S^i, q_2 = \sum_{i=0}^{\infty} q_i^2 S^i,\\
q_4 &= \sum_{i=0}^{\infty} q_i^4 S^i, q_5 = \sum_{i=0}^{\infty} q_i^5 S^i,
\end{aligned}
\tag{1.8.12}
$$

where the coefficients q_i^j, $j=1,2,4,5,\ i=0,1,2,...$, are expressed as

$$
\begin{aligned}
q_i^1 &= \frac{1}{i}\left[\lambda\left(\frac{1}{c}-\frac{1}{a}\right)q_{i-1}^2+\left(1+\frac{1}{e}-\frac{1}{g}\right)q_{i-1}^5\right],\\
q_i^2 &= \frac{1}{i}\left[\lambda\left(\frac{1}{b}-\frac{1}{c}\right)q_{i-1}^1+\left(\frac{1}{g}-1-\frac{1}{f}\right)q_{i-1}^4\right],\\
q_{i-1}^4 &= \frac{1}{i}\left[\frac{q_{i-1}^2}{a}+\lambda\frac{q_{i-1}^5}{c}\right],\\
q_{i-1}^5 &= \frac{1}{i}\left[-\frac{q_{i-1}^1}{b}-\lambda\frac{q_{i-1}^4}{c}\right].
\end{aligned}
\tag{1.8.13}
$$

Obviously $q_1(0)=q_0^1=0$, $q_2(0)=q_0^2=0$, further, it could be assumed that $q_0^4=1$ while λ_{cr} determined from (1.8.9) and the coefficient q_0^5 should satisfy

$$
q_1(1)=\sum_{i=1}^{\infty} q_i^1=0, \qquad q_2(1)=\sum_{i=1}^{\infty} q_i^2=0. \tag{1.8.14}
$$

By numerical examination of the determined coefficients it could be shown that each radius of convergence of the series (1.8.12) is greater than 1. To show that the critical load parameter $\lambda = \lambda_{cr}$ is not multiple zero of the system we use the same argument as before. Also, λ_{cr} determines the coefficients (1.8.13) uniquely so the eigenvector of the adjoint system (1.8.11) could be written as

$$q_s = D \begin{bmatrix} \sum\limits_{i=0}^{\infty} q_i^1 S^i \\ \\ \sum\limits_{i=0}^{\infty} q_i^2 S^i \\ \\ \sum\limits_{i=0}^{\infty} q_i^4 S^i \\ \\ \sum\limits_{i=0}^{\infty} q_i^5 S^i \end{bmatrix}, \tag{1.8.15}$$

where the constant D could be chosen so that $\|q_s\|_2 = 1$.

The preceding results lead to the conclusion that the dimension of the null space of $\mathcal{L}(\lambda_{cr})$ and the codimension of the range space of $\mathcal{L}(\lambda_{cr})$ are equal to 1, that is

$$\dim N(\mathcal{L}(\lambda_{cr})) = co\dim R(\mathcal{L}(\lambda_{cr})) = 1.$$

The last relation states that the algebraic and geometrical multiplicity of the eigenvalue of linearized operator, λ_{cr}, equals unity, i.e., the eigenvalue of the linearized operator is a simple one. In other words, the nonlinear operator $\mathcal{F}$ is the Fredholm operator with index zero.

Now, when all the necessary preparations were done the main result of the stability analysis concerning the existence of the nontrivial equilibrium configuration, could be stated as follows:

Theorem 1 *The bifurcation points of the nonlinear system (1.8.1) are given in the form* $(\lambda_{cr}, 0)$ *where* λ_{cr} *represents the solution of (1.8.9).*

Proof. The main idea is to transform the nonlinear equation (1.8.1) into the form to which well developed techniques of the nonlinear analysis could be applied easily. We may proceed as follows. First rewrite (1.8.1) in the form

$$\mathcal{F}(\lambda, u) = \mathcal{G}u - \lambda\mathcal{H}u + \mathcal{N}(\lambda, u),$$

where $\mathcal{N}(\lambda, u)$ is as in (1.8.1) and where

$$\mathcal{G} = \frac{d}{dS} + \begin{bmatrix} 0 & 0 & 0 & -\frac{1}{b} \\ 0 & 0 & \frac{1}{a} & 0 \\ 0 & \frac{1}{g} - \frac{1}{f} - 1 & 0 & 0 \\ 1 + \frac{1}{e} - \frac{1}{g} & 0 & 0 & 0 \end{bmatrix},$$

and

$$\mathcal{H} = \begin{bmatrix} 0 & \frac{1}{c} - \frac{1}{b} & 0 & 0 \\ \frac{1}{a} - \frac{1}{c} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{c} \\ 0 & 0 & -\frac{1}{c} & 0 \end{bmatrix},$$

are bounded linear operators from H_0^1 into Q. Then, note that $\mathcal{N}(\lambda, 0) = 0$ and $\mathcal{D}_u\mathcal{N}(\lambda, 0) = 0$ where $\mathcal{D}_u\mathcal{N}(\lambda, 0)$ denotes the Fréchet derivative of $\mathcal{N}$ at $u = 0$. Finally, we recall that the solutions of the corresponding linear spectral problem are geometrically simple. The preparation is finished and all the assumptions of the Theorem 5.4 for the case $N = 1$ of Chow and Hale (1982., p.191) are recognized, so the statement of the theorem follows.

Hence, it was shown that the bifurcation points of the nonlinear equations are determined by the bifurcation points of the linearized equations. With this existence result we may proceed to numerical analysis of the rod problem.

2.9 Determination of the critical bifurcation parameter and the post-critical shape of the column

In this section numerical solutions of the system (1.8.9) as well as results of numerical integrations of (1.7.4) are presented. In numerical analysis of both problems Newton's method was applied as it was recommended in Press *et al.* (1986). In order to examine the influence of shear and compressibility on the critical load and the postbuckling behavior of compressed and twisted rod, several characteristic values in the space of physical parameters are chosen. Since the dimension of that space is relatively high, as could be seen at least from (1.7.1), the usual graphical and tabular presentations which cover intervals of dimensionless parameters will be avoided. Because of the fact that personal computers are in rather wide use as well as the relative simplicity of the system (1.8.9) results for vast interval of parameters could be obtained readily. To illustrate the exposed material the rod with arbitrary selected parameters, partly on the literature, partly under assumption that "there exist material whose parameters are ...", will be considered. Results of numerical experiments could be presented more easily if we add two dimensionless parameters

$$\rho^2 = \frac{L^2 A}{I}, \qquad \kappa = \frac{P}{P_E}, \quad (0 < k < 1)\,,$$

to (1.7.1). Namely, ρ represents slenderness ratio, (A and I are the area and moment of inertia of the cross-section respectively), while κ represents which part of the Euler buckling load, denoted by P_E, compresses the considered rod. The Euler buckling load (the compressing force) for the case of hinged rod equals $P_E = \pi^2 EJ/L^2$, see Atanackovic (1987, p.67). The values $\kappa > 1$ allow the buckling in advance so will not be considered here. As recommended in Biezeno and Grammel (1953) a real load of the compressed and twisted column is in the area of a relatively small value of the twisting couple and the compressing force not far from P_E. For slender rods the usual values for ρ are near 100, but in some references a class of so-called stocky structures with $\rho \leq 10$ is sometimes considered.

Numerical experiments were performed in the following way. First, κ was chosen, then b was determined as $\left(\kappa\pi^2\right)^{-1}$, then the values for a and c were chosen. The extensional rigidity g was determined as $b\rho^2$. The shear rigidities f and h are chosen to be $0.3g$ and $0.2g$ respectively. The last choice was motivated by the work of Renton (1991). For the infinite values of shear and extensional rigidities the results of Spasic (1991) were recovered. These values were used as the initial approximation for the solution of the equations corresponding to the finite values of EA (or g) and GA (or e, f).

In Table 1 the critical buckling load λ_{cr} is given for three types of rod cross-sections and for several values of the parameters κ, ρ, e, f, and g. Although it was not specially noted, in the last column of Table 1, which corresponds to equal values of a and b, the equal values of e and f were taken. In that column for infinite values of e, f and g the results of Grammel (1923) and Voljmir (1967, p.162) were recognized. The critical load parameter λ_{cr} shown in Table 1 satisfies the equations (1.8.14) and the conditions which exclude multiplicity of zeros of the system (1.8.9).

Table 1. Critical buckling load.

Rod description					λ_{cr}		
					$b = (\kappa\pi^2)^{-1}$		
					$a = 2b$	$a = 2b$	$a = b$
κ	ρ	e	f	g	$c = 2b/3$	$c = 3b/2$	$c = 2b/3$
2/3	100	$\to\infty$	$\to\infty$	$\to\infty$	0.70632	0.52222	0.55133
		$\to\infty$	$\to\infty$	1520	0.70733	0.52257	0.55169
		456	304	$\to\infty$	0.70140	0.52121	0.55012
		456	304	1520	0.70241	0.52156	0.55048
	10	$\to\infty$	$\to\infty$	15.2	0.79801	0.55594	0.58648
		4.56	3.04	$\to\infty$	0.36357	0.39879	0.41307
		4.56	3.04	15.2	0.41837	0.44403	0.45894
19/20	100	$\to\infty$	$\to\infty$	$\to\infty$	0.13543	0.14121	0.14984
		$\to\infty$	$\to\infty$	1066	0.13672	0.14246	0.15117
		320	213	$\to\infty$	0.13074	0.13700	0.14533
		320	213	1066	0.13205	0.13829	0.14670
	10	$\to\infty$	$\to\infty$	66.7	0.15511	0.16011	0.16986
		20	13.3	$\to\infty$	0.02825	0.03169	0.03347
		20	13.3	66.7	0.07380	0.08216	0.08672

The results shown in Table 1 lead to conclusion that the shear rigidities (e, f or kGA, hGA) and the extensional rigidity (g or EA) have the opposite influence on the value of the critical load parameter λ_{cr}. Namely, decreasing the value of shear rigidities decreases the value of λ_{cr} and decreasing the value of extensional rigidity the value of λ_{cr} increases. This conclusion represents the generalization of the corresponding result of generalized plane elastica theory, see Atanackovic and Spasic (1991).

Also, in the case of slender rods ($\rho = 100$) the critical load for the classical Kirchhoff model and for the rod with finite values of shear and extensional rigidities, proposed by Eliseyev, are almost the same, but in the case of the stocky structures ($\rho = 10$) this is not the case. The finite values of the shear and extensional rigidities in the case of stocky structures do influence on the critical load. This result is also seen in the generalized plane elastica theory, see Goto *et al.* (1990, p. 385).

By comparison the critical buckling load presented in the last three columns the influence of the rod cross-section as a function of κ is analyzed. The property that for the small values of κ increasing a, for fixed b and c, increases λ_{cr} and for κ near 1 the opposite is valid, revealed in Spasic (1991) where infinite values of e, f and g are considered, appeared again for the case of slender rods and finite values of e, f and g.

To analyze the postcritical shape of the column two-point boundary value problem (1.7.4) was solved for two different models. In both cases the results presented in Table 1 were used. First, the rod with $\kappa = 19/20$, $b = 0.1066$, $a = 2b$ and $c = 2b/3$ was chosen. In order to compare the Kirchhoff and the model of Eliseyev for rigidities e, f and g infinite values, and the values 310, 210 and 1100 were chosen respectively. As the second, the rod with e, f and g equal 20, 13.33 and 66.66 respectively was chosen. For the load parameter $\lambda > \lambda_{cr}$ the value 0.16 was selected. The problem was solved by the shooting method. The values $\psi(0)$ and $\vartheta(0)$ were assumed. The reduction of the boundary value problem to the Cauchy problem with the same solution was done by the Newton method. In Press *et al.* (1986) it was emphasized that the success of the shooting procedure depends on the step used for calculation of the partial derivatives needed. For the solution of (1.7.4) that step was chosen as it was recommended in Atanackovic and Djukic (1992). All the numerical integrations were based on the Bulirisch-Stoer method. In Fig. 2a the trivial configuration of the rod with $\kappa = 19/20$, $b = 0.1066$, $a = 2b = 3c$, $e = 310$, $f = 210$ and $g = 1100$ for $\lambda = 0.13 < \lambda_{cr}$ was shown. In Fig. 2b the nontrivial configuration of the same rod loaded with $\lambda = 0.16 > \lambda_{cr}$ was shown. The different lines were used to mark four edges of the rod which were parallel in the undeformed state (with Q as a rectangular plate). Besides the rod, in Fig. 2b the corresponding orthogonal projections of the rod axes were also shown. The corresponding maximal values of the geometrical quantities considered were shown in Table 2.

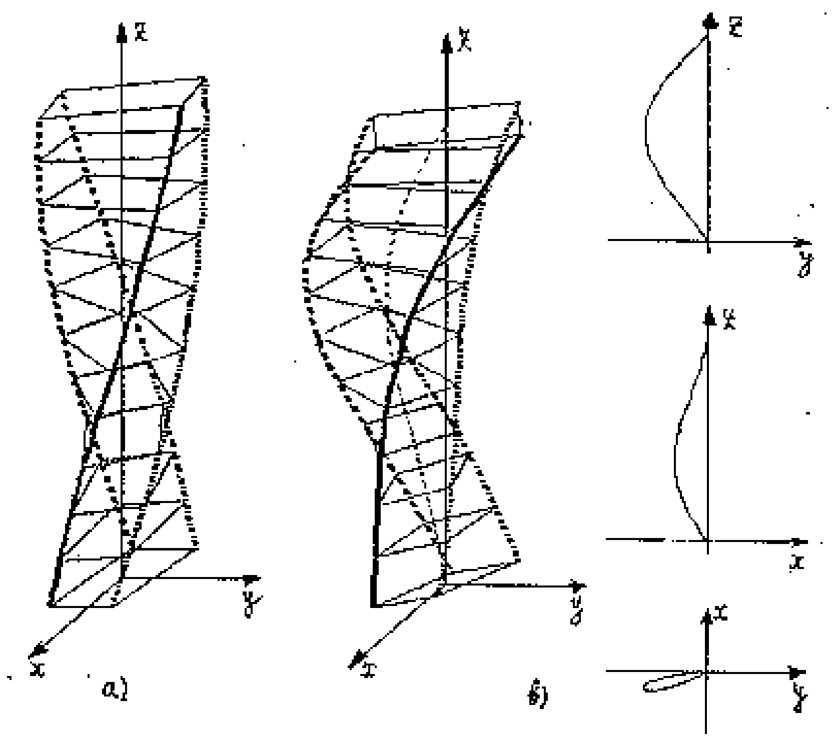


Fig. 2. Trivial and nontrivial configuration of the compressed and twisted rod with shear and axial strain.

The same experiment was repeated for the rod with $\kappa = 2/3$, $b = 0.152$, $a = 2b = 3c$.

The Kirchhoff model ($e,\ f,\ g \to \infty$) was compared to the Eliseyev model for the case of the slender rod ($\rho = 100$, $e = 460$, $f = 300$ and $g = 1500$) and for the stocky rod ($\rho = 10$, $e = 4.6$, $f = 3.0$ and $g = 15.0$). For $\lambda > \lambda_{cr}$ the value 0.708 was chosen. The lists of results that could be used for detailed comparison of all relevant variables are given in Subsection 5.2. As before in Table 2 only the maximal values of characteristic variables are given, (the values in the parenthesis represents the corresponding dimensionless arc length S).

Table 2. Maximal values of the geometrical variables on the solutions of (1.7.4).

	$\kappa = 19/20$			$\kappa = 2/3$		
	$\lambda = 0.16$			$\lambda = 0.708$		
	$a = 0.2133,\ b = 0.1067,\ c = 0.0711$			$a = 0.304,\ b = 0.152,\ c = 0.1013$		
e	$\to \infty$	320	20	$\to \infty$	460	4.6
f	$\to \infty$	210	13	$\to \infty$	300	3.0
g	$\to \infty$	1100	66	$\to \infty$	1500	15.2
x	-0.05411 (0.4)	-0.05737 (0.4)	-0.08829 (0.4)	0.26548 (0.6)	0.26619 (0.6)	0.31380 (0.6)
y	-0.10392 (0.5)	-0.11026 (0.5)	-0.17322 (0.5)	-0.17702 (0.3)	-0.17773 (0.3)	-0.21506 (0.3)
z	0.96479 (1)	0.95935 (1)	0.88413 (1)	0.61412 (1)	0.61040 (1)	0.32084 (1)
ψ	-0.27028 (0)	-0.28734 (0)	-0.45995 (0)	-1.01785 (0.9)	-1.02070 (0.9)	-1.13828 (0.9)
ϑ	-0.37568 (1)	-0.39825 (1)	-0.61700 (1)	1.06710 (0)	1.07032 (0)	1.21274 (0)
φ	2.20211 (1)	2.19645 (1)	2.13184 (1)	5.10330 (1)	5.09686 (1)	4.82405 (1)
α	0	-0.00090 (0.2)	-0.02260 (0.2)	0	-0.00039 (0.3)	-0.06346 (0.3)
β	0	0.00128 (0)	0.02864 (0)	0	0.00292 (0)	0.30060 (0)
ε	0	-0.00091 (0.5)	-0.01485 (0.5)	0	-0.00049 (0.5)	-0.02411 (0.5)
$\mathbf{M}$	0.19718 (0.5)	0.20136 (0.5)	0.24950 (0.5)	0.76304 (0.5)	0.76335 (0.5)	0.78512 (0.5)

The examination of the numerical results presented in Table 2 leads to the conclusion that the predictions of the postcritical shape for both the Kirchhoff and the Eliseyev model, for the case of slender rod, are almost the same. The difference in geometry in postbuckling region becomes more significant in the case of the nonslender rods. In such a case the values of α, β and ε are not to be neglected, i.e., the effects of shear and compressibility should be taken into account in the buckling analysis. The consideration of this so called stocky structures is the subject of investigation of many papers.

Note that besides the trivial configuration two connected nontrivial configurations were found. Namely, for each solution determined by $\psi(0)$ and $\vartheta(0)$ there is also the solution determined by -$\psi(0)$ and -$\vartheta(0)$. When it was possible, the integration procedure was controlled by the first integral (1.7.9). For example for the case $\lambda = 0.16$, $a = 2b = 3c$, e, f, $g \to \infty$ the integral (1.7.9) at the trivial solution is -1.63996 and at the nontrivial solution is -1.52321. The first integrals (1.7.8) have the same value at both trivial and nontrivial solutions as expected.

The study of bifurcation of the trivial solution of a compressed and twisted rod will be continued in the forth section of this thesis.

3 Optimal shape of the rod against buckling

3.1 Preliminaries

In 1883 Grenhhill studied buckling of a column under terminal thrust and torsion. For the given terminal torque he determined the Euler buckling load for the column of uniform circular cross-section. Roughly speaking this part deals with a generalization of that problem in the following sense. For a column of given length, volume and twisting torque, find the shape which will give the largest possible compressing force as the buckling load. For the zero torque this problem reduces to the well known problem posed by Lagrange in 1773, see Banichuk (1980).

In what follows we shall recall Pearson's formulation of the Lagrange problem, that is, *to find the curve which by its revolution about an axis in its plane determines the column of great efficiency,* see Cox and Overton (1992). History of the problem goes back to 1759. Namely, in his papers of that time, Euler examined critical load for the compressed columns of variable cross-sections, (shape like a segment of a cone, paraboloidal shape etc.), see Nikolai (1928). The solution of the posed problem was subject of investigation of many authors, so there is voluminous literature that could be related to it. Even today the interest for the problem is still significant, among others see Keller (1961), Tadjbakhsh and Keller (1962), Troicki and Petuhov (1982), Seiranian (1984), Bratus (1986), Barnes (1985, 1988), Bratus and Zharov (1990) and Cox and Overton (1992).

As in the previous section the strategy for the spatial problem will be motivated by the plane elastica theory. Also it is determined by the fact that the procedure of problem formulation in the optimal control theory is not unique, see Alekseev et al. (1979). In general, the same problem allows several variational as well as several formulations belonging to modern optimal control theory. The choice and corresponding tools, represent a separate problem which will determine the success in problem solving procedure. In order to motivate the approach to be followed, let us recall formulations presented in papers of Chung and Zung (1979) and Bratus and Zharov (1990), where optimal shape of a compressed column, free at one, and clamped at the other end, is considered. In the first paper, on the basis of the linearized equations, necessary conditions for optimal control problems were derived by use of the maximum principle of Pontryagin. In the second paper, where the finite deformations of the column with respect to the trivial equilibrium configuration, were allowed, authors use variational calculus with nonlinear equilibrium equations. For the case of small deformations both solutions coincide as expected. These two papers reveal the strategy needed for the problem which distribution of material along the length of compressed and twisted column will give the column of minimum volume and will support a given load without spatial buckling.

Let us consider a slender column represented in rectangular Cartesian coordinate system Ozx by a plane curve C of unit length. The curve C represents the column axis which coincides with the centroidal line of the column cross-sections. It is assumed that the column is hinged at either end, the hinge at the origin O being fixed whereas the other one is free to move along the axis z. The column is loaded by a concentrated force P retaining the action line along the z axis which coincide with the column axis in the undeformed state. We denote the bending rigidity of the column and the angle between the tangent to the column axis and the z axis with $EI = EI(S)$ and $\psi = \psi(S)$ respectively. Here S denotes the arc length of C measured from one end point. Implicitly it is assumed here that extensional and shear rigidity are infinite, for now there is no torque at the end. We note that according to Pearson's formulation, the column is of circular cross section, of area $A = A(S)$, so that either $EI(S)$ or $A(S)$ determines the distribution of material along the length of a column. In the following we shall analyze only the first (symmetrical) buckling mode, having maximal value of the bending moment at the middle, and thus consider only half of the column. This fact has a considerable advantages since the interval for numerical integration procedures is halved.

The half volume of the column reads

$$V = \int_0^{1/2} \sqrt{EI(S)}dS, \tag{2.1.1}$$

(note $EI = EA^2/(4\pi)$, and the constant was omitted). The differential equations describing the equilibrium configuration of the column in the dimensional form, for the plane problem and the classical Bernoulli-Euler elastica theory are the special case of the equations (1.7.4), (1.7.1). With respect to the version of Euler angles used, the linearized equations are easily obtained. Instead of the boundary condition at the right end the boundary condition corresponding to the first (symmetrical) mode $\psi(1/2) = 0$ will be used. Now, following the lines of the above papers the column's resistance to buckling under axial compression, as a problem of optimal control theory, could be expressed, at least in three different ways:

I) to find the distribution of material EI along the length of a column so that the column is of minimum volume and will support a given load without buckling, in our notation, reads

$$\min_{EI} V,$$

$$\dot{x} = \psi, \quad \dot{\psi} = -\frac{Px}{EI},$$

$$x(0) = 0, \quad \psi(1/2) = 0,$$

II) to find the distribution of material EI along the length of a column of a given volume V_p, which will give the largest possible buckling load, i.e.

$$\max_{EI} P,$$

$$\dot{x} = \psi, \quad \dot{\psi} = -\frac{Px}{EI}, \quad \dot{V} = \sqrt{EI}, \quad \dot{P} = 0,$$

$$x(0) = 0, \quad \psi(1/2) = 0, \quad V(0) = 0, \quad V(1/2) = V_p,$$

and

III) to find the distribution of material EI along the length of a column of a given volume V_p, which will give the largest possible load provided given post-buckling deformation x_p will not be exceeded, i.e.

$$\max_{EI} P,$$

$$\dot{x} = \sin\psi, \quad \dot{\psi} = -\frac{Px}{EI}, \quad \dot{V} = \sqrt{EI}, \quad \dot{P} = 0,$$

$$x(0) = 0, \quad \psi(1/2) = 0, \quad V(0) = 0, \quad x(1/2) = x_p, \quad V(1/2) = V_p,$$

with x_p as the maximum deflection of the column in the post-critical region. Since x_p is not restricted to be small the nonlinear equilibrium equations are to be used. In both second and third formulation isoperimetric problems were reformulated by introducing new differential equations, as usual, see Alekseev et al. (1979). Also note that for small values of x_p, corresponding to small values of ψ, the second and third formulation coincide.

According to Pontryagin's maximum principle, necessary conditions for optimality, the Euler-Lagrange or costate equations, and natural boundary conditions for these three problems read

I)

$$EI = (-2Pxp_2)^{2/3},$$

$$\dot{p}_1 = \frac{Pp_2}{EI}, \quad \dot{p}_2 = -p_1,$$

$$p_1(1/2) = 0, \quad p_2(0) = 0,$$

II)

$$EI = \left(\frac{-2Pxp_2}{p_3}\right)^{2/3},$$

$$\dot{p}_1 = \frac{Pp_2}{EI}, \quad \dot{p}_2 = -p_1, \quad \dot{p}_3 = 0, \quad \dot{p}_4 = \frac{p_2 x}{EI},$$

$$p_1(1/2) = 0, \quad p_2(0) = 0, \quad p_4(0) = 0, \quad p_4(1/2) = 1,$$

III)

$$EI = \left(\frac{-2Pxp_2}{p_3}\right)^{2/3},$$

$$\dot{p}_1 = \frac{Pp_2}{EI}, \quad \dot{p}_2 = -p_1 \cos\psi, \quad \dot{p}_3 = 0, \quad \dot{p}_4 = \frac{p_2 x}{EI},$$

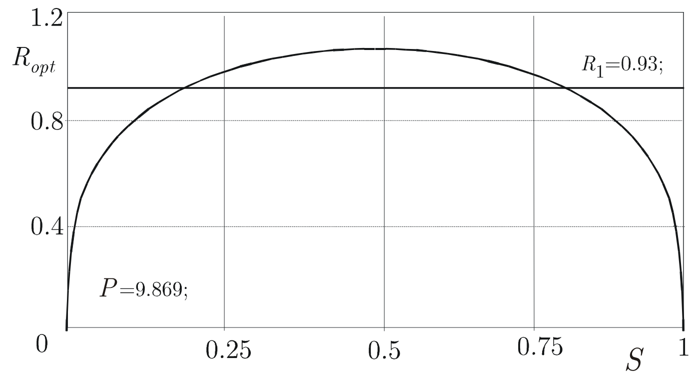
$$p_2(0) = 0, \quad p_4(0) = 0, \quad p_4(1/2) = 1,$$

where the Lagrange multipliers p_1, p_2, p_3 and p_4 and the corresponding Hamiltonians are introduced in the usual way.

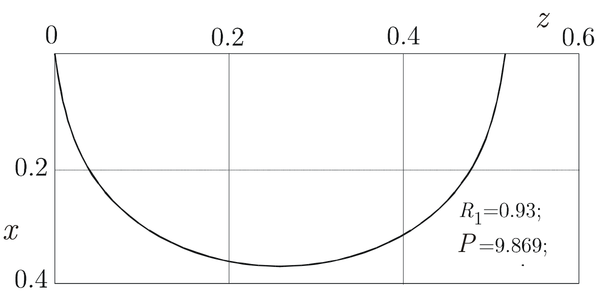
As in Section 1.9 boundary value problems are solved numerically by reducing to initial value problems with the same solutions (a well known shooting method). Numerical experiments were done for several values of the parameters P, x_p, V_P. We show one solution of problem I) and compare it with solutions of alternative problems II) and III) for the specially chosen values of parameters.

A slender cylindrical column of unit length and unite volume (i.e. with uniform distribution of the material along the column axis), will buckle for the critical load $P = \pi^2 = 9.869$, see Atanacković (1986). If one put this value in problem I) than the obtained minimum volume of the optimal column is $V_{\min} = 0.866$. In Figure 3a the optimal shape, as a solution of the optimal control problem is presented. Instead of EI we present the optimal curve obtained as $R_{opt}(S) = \sqrt[4]{EI_{opt}(S)}$, which according to Pearson formulation, gives the column of greatest efficiency. The obtained shape corresponds to Clausen's solution of the Lagrange problem. The uniform column of the same volume as the optimal one, with corresponding constant radius of the cross-section $R_1 = \sqrt[4]{EI_1} = \sqrt[4]{0.75} = 0.93 < 1$, is also shown.

In order to examine the postbuckling behavior of the column of unit length and the volume $V_1 = 0.866$ with uniform cross section first, we note that its buckling load is $P_1 = 7.402 < \pi^2$. The optimal column of the same volume, loaded with P_1 still remains straight. In Figure 3b the post-critical shape of the curve C for the uniform column with the same volume as the optimal is shown. By solving two point boundary value problem describing equilibrium configuration of the uniform column with $P = \pi^2$ and $EI_1 = 0.75$, we note very large deflection of the column axis. In other words, when loaded with $P = \pi^2$ the optimal column is about to begin to buckle while the uniform column of the same volume is far in the post-buckling region. As a result of the optimization procedure we conclude that the optimal column, that requires a special manufacturing technology, will save about 13.4% of material and will increase the Euler buckling load for about 26%.



a)



b)

Fig. 3. Clausen's solution and uniform column of the same volume (a) and postcritical shape of the uniform column, (b) loaded with $P = \pi^2$.

It may be added that the optimal shapes for the first and the second formulation are the same. Namely, by choosing $V_p = 0.433$ and solving the second problem one gets $P_{\max} = \pi^2$ as expected. Also, if we put the value that corresponds to the column of unit length and unite volume, $V_p = 0.5$, from the second problem we obtain $P_{\max} = 13.159$. Similarly, with value $P = 13.159$ according to the first formulation the minimum half volume of the optimal column equals $V_{\min} = 0.5$. The first and the second formulation are equivalent, see Troicki and Petuhov (1982) or Bratus and Zharov (1990), but the first one is easier to tackle. Namely, in the first case one has to shoot a point in $\mathbb{R}^2$ i.e. $\psi(0), p_1(0)$ while for the second the shooting point is in $\mathbb{R}^4$, that is $\psi(0), p_1(0), P, p_3(0)$. Another problem is connected with uniqueness of the solution of two point BVP. For example the same solution is obtained when started at $S = 0$ with either $(0.378, -29.004, 13.159, -52.638)$ or $(0.216, 485.626, 13.159, 52.638)$.

To investigate the agreement of the second and the third solution first we need to examine the value x_p. In Bratus and Zharov (1990), where instead of $x_p = x(1/2)$ the value $\psi(0)$ is proposed, very large deflections in post buckling region are allowed. Some of them are even bigger then the half length of the column. A first observation is that in engineering we intend to keep column in trivial - almost straight position so we propose here the values of x_p to be not more than 10% of the column length. If we solve the third problem with $V_p = 0.5$ and $x_p = 0.102$ we get $P_{\max} = 13.425$. As expected, decreasing of x_p the closeness of the second and the third solution increases. For example, with $V_p = 0.5$ and $x_p = 0.05$ we get $P_{\max} = 13.221$. In such a case we find that the difference between solution for $EI_{opt}(S)$ obtained from the second (linear) problem, and the corresponding solution of the third (nonlinear) problem, is less then 10^{-2}.

In conclusion we claim that among the equivalent problems the first one is optimal for engineering applications. Namely, the previous analysis shows that the first formulation, based on based linear equilibrium equations, corresponds to the problem of minimum dimension and is most tractable with respect to numerical analysis.

Finally, we give two possible generalizations of the above optimal design problems.

Analytical solution of the Lagrange problem given in framework of variational calculus by Clausen in 1851, for some authors was considered as incomplete since the obtained optimal shape did have end points where the cross section vanishes. It means that in the neighborhood of its ends the optimal column does not recognize the difference between load P and for example its doubled value $2P$, see Fig. 3a. In 1907, Nikolai, was the first author who considered that anomaly of Clausen's solution. In order to avoid any finite load to induce infinite stresses in the column, Nikolai proposed minimal cross sectional area at the ends, determined so that given limiting stress will not be exceeded. It should be noted that Nikolai's solution goes beyond the framework of the classical Bernoulli-Euler elastica theory, see Nikolai (1928). In the opinion of the author of this thesis, Clausen's solution is mathematically correct when considered in the framework of the classical elastica. Namely, in the Bernoulli-Euler elastica theory the compressibility of the column axis is neglected. As a consequence, in the trivial equilibrium configuration forces P and $2P$ have the same effect on the column axis. In order to recognize the load, a new condition introduced by Nikolai, takes into account only compressibility. It is a well known fact that compressibility and shear have the opposite influence on the Euler buckling load so it is a question of interest to examine both effects on the optimal shape. In other words, we pose a question does optimal column with finite values of extensional and shear rigidity lead to non vanishing cross-sections at its ends. The answer to that question should be considered in the sense of the generalized elastica with shear and compressibility, see Atanacković and Spasić (1993).

Another generalization that could be related to the problem of Lagrange is connected with constrains on control variable and is motivated by the fact that modern design requires limited dimensions ab initio. For the above problems it means that the condition $EI_{\min} \leq EI(S) \leq EI_{\max}$ should be involved. With this type of constraint it is natural to formulate the optimal design problem in the framework of Pontryagin maximum principle since it is more general then the classical variational calculus. With a condition of this type the singularity of the coefficient function in differential equilibrium equations is eliminated, see Barnes (1988), but numerical aspects of the problem become more complicated. In such a case, an interesting approach for solution could be found in paper of Krilov and Chernousko (1972). In applications, optimal control problems where convexity assumption for the control domain is not needed are mathematically attractive as well as technically significant, see Nagahisa and Sakawa (1991). Roughly speaking, in the problem of Lagrange we could propose the strongest column to be made of only a few circular cross sections of different size, i.e., in our notation it means to impose constraint on bending rigidity $EI \in \{EI_1, ..., EI_l\}$, for a few given real numbers $EI_1, ..., EI_l$. This could reduce the column weight as well as the expenses of column production. In this case also, as in the previous one, it is natural to formulate the problem by use of the maximum principle.

3.2 Optimal shape of the column against spatial buckling

In order to pose the generalized Greenhill's problem, in the sense as shown in the opening we recall some of the results obtained in Sections 1.8 and 1.9. Namely, the bifurcation theorem (Theorem 1 of 1.8), and numerical results presented in Table 1 of 1.9, lead to the following. First, the critical load of the compressed and twisted column is determined by the linearized equilibrium equations, and second, for slender columns (with slenderness ratio $\rho \approx 100$) the effects of shear and compressibility could be neglected. These facts lead to significant simplification of the equilibrium equations that will be used as a model in optimal control problem. The formulation of the first type, Pearson's formulation and the assumption of unit length make the model more tractable.

The equations of the model follows from (1.7.4) with $e, f, g \to \infty$, the values that correspond to the classical Kirchhoff-Clebsh theory, as well as with $a = b$, $c = a/(1+\nu)$, ν being Poisson's coefficient equals 0.3 in all numerical experiments of this section. With the proposed values of the parameters the equilibrium equations should be linearized in the neighborhood of the trivial equilibrium configuration. The simplicity of such procedure is ensured by the version of Euler angles used, see Section 1.5. These angles are suitable also for columns with different shape of the cross-section since it allows an easy linearization in the case when ψ and ϑ are small and φ being of finite amount. After linearization the equation $(1.7.4)_3$ becomes formal identity $z = S$ and will be omitted. Finally, by use of (1.7.1), the obtained equations will be given in full dimensional form.

From the linearized equilibrium equations, critical force P and twisting couple W, for the column of constant cross-section, unit volume and unit length, are determined from

$$\frac{W^2}{4} + P - \pi^2 = 0, \tag{2.2.1}$$

see Grammel (1923) and Atanacković (1986, pp. 133). As suggested in the previous section we shall reduce the integration interval by substituting the conditions at the end $S = 1$ with the ones corresponding to $S = 1/2$. Namely, as before we consider only the symmetrical buckled mode, in which observers from both ends A and B, see the centroidal line (column axis) in the same way, see Fig. 4. In such a case instead of $y(1) = 0$ from (1.7.4) we use the condition $y(1/2) = 0$, (referring to Fig. 4 again we find $x(S) = x_1(S_1)$,

$y(S) = -y(S_1)$, $S = 1 - S_1$ and the condition follows).

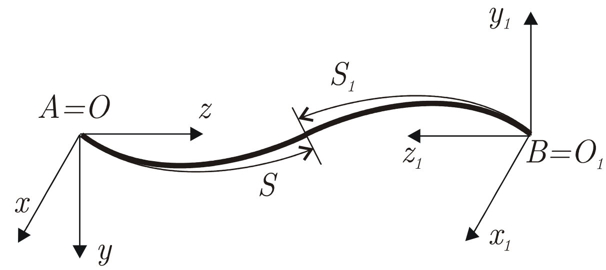


Fig. 4. Two coordinate systems.

The second property of the first (symmetrical) mode is the maximal value of the bending moment at the middle. In the sense of classical elastica this fact together with $(1.7.2)_{4,5,6}$ and $(1.7.4)_{1,2}$ allows the substitution $\psi(1/2) = 0$ instead of $x(1) = 0$ from (1.7.4).

With this preparation done we may pose the following:

Problem *to find the distribution of material EI along the length of a column, so that the column is of minimum volume and will support a load for given P and W determined from (2.2.1), without buckling, i.e.,*

$$\min_{EI} V,$$

$$\dot{x} = \psi, \quad \dot{y} = -\theta, \quad \dot{\psi} = \frac{W\vartheta - Px}{EI}, \quad \dot{\vartheta} = \frac{Py - W\psi}{EI}, \quad \dot{\varphi} = \frac{(1+\nu)W}{EI}, \tag{2.2.2}$$

$$x(0) = 0, \quad y(0) = 0, \quad \varphi(0) = 0, \quad y(1/2) = 0, \quad \psi(1/2) = 0.$$

According to Pontryagin's maximum principle, nesessary optimality conditions read

$$EI = \{2[p_3(W\vartheta - Px) + p_4(Py - W\psi) + p_5(1+\nu)W]\}^{2/3}$$

$$\dot{p}_1 = \frac{Pp_3}{EI}, \quad \dot{p}_2 = -\frac{Pp_4}{EI}, \quad \dot{p}_3 = -p_1 + \frac{Wp_4}{EI}, \quad \dot{p}_4 = p_2 - \frac{Wp_3}{EI}, \quad \dot{p}_5 = 0. \tag{2.2.3}$$

$$p_1(1/2) = 0, \quad , p_3(0) = 0, \quad p_4(0) = 0, \quad p_4(1/2) = 0,$$

and

$$p_\varphi(1/2) = 0, \tag{2.2.4}$$

since the value of the twisting angle at the middle of the column $\varphi(1/2)$ is not specified.

From (2.2.3) and (2.2.4) it follows that on the optimal shape bending rigidity EI at $S = 0$, equals zero, and is very small in the neighborhood of the point $S = 0$ because of continuity. For small values of EI any kind of activity related to numerical solution of the system (2.2.2), (2.2.3) and (2.2.4) becomes very complicated since the stiff equation problem appears, see Press et al. (1986). The situation will be even more complicated if we had chosen to generalize problem III) with nonlinear differential equations as a model. We note that some stiff equations can be handled by a change of variables, see Acton (1970), but we shall avoid that problem here on the basis of physical considerations.

Namely, if in addition to boundary conditions (2.2.2) we propose the value of the twist angle at the middle of the column, say

$$\varphi\,(1/2) = \varphi_p, \tag{2.2.5}$$

then for the obtained optimal control problem, from the transversality conditions, the value of generalized impulse p_5 becomes unspecified. Therefore, by substituting (2.2.4) with (2.2.4) the value of p_5 does not have to vanish at $S = 0$. In such a case the value of $EI\,(0)$ will also differ from zero.

The next problem to be solved is the value of φ_p . Namely, for the uniform (cylindrical) column of unit volume and length in the trivial equilibrium configuration, in which the column is straight but twisted, the twist angle at the middle reads

$$\varphi_1 = \frac{(1+\nu)\,W}{2}. \tag{2.2.6}$$

In the formulation of the optimization problem we expect to get the volume of optimal column to be less then one, so the value of the twisting angle that corresponds the uniform column of the same volume as optimal, say $\varphi_k = \varphi\,(1/2)$ will be greater then φ_1. Thus, we propose φ_p to be greater then φ_1. As a result we expect the values φ_p and φ_k to be close, and that the value of the material savings to be in correlation with the difference $\varphi_p - \varphi_k$. Also, for the very small values of the twisting couple W we expect the optimal shape of the compressed and twisted column to be very close to the Clausen's solution presented in Figure 3.

The formulation of the optimal control problem is done, the optimality conditions are shown above, and thus the main result of this section in the following form

Theorem 2 *Compressed and twisted column of greatest efficiency is determined by the solution of system (2.2.2), (2.2.3) and (2.2.5),*

has been proved.

It remains to solve the problem numerically.

3.3 Numerical results

For different values of the force P, for W determined from (2.2.1) and for several values of φ_p, boundary value problem (2.2.2), (2.2.3) (2.2.5) was solved. As in the case of (1.7.4) the shooting method is used. Initial guess $\psi\,(0)$, $\theta\,(0)$, $p_\xi\,(0)$, $p_\eta\,(0)$ and $p_\varphi\,(0)$ was improved by Newton method. Numerical solution was identified when either sum of the absolute values of functions included in the conditions at $S = 1/2$ becomes less then 10^{-5} or sum of the absolute values of changes in initial conditions becomes less then 2.5×10^{-4}. It could be easily shown that in the case of formulation that corresponds to II) of Section 2.1, seven initial values are to be guessed, representing more difficult numerical problem.

All numerical experiments are done in the area of small W and P near π^2 as suggested by Biezeno and Grammel (1953). Namely, the real compressed and twisted columns are loaded near Euler buckling load $P = \pi^2$ and small value of twisting couple. In all the numerical calculations the value $\nu = 0.3$ is used. As expected the bending rigidity of the column differs from zero for all values of $S \in [0, 1/2]$. Another expected fact that the values of P near π^2 and W near zero will give the optimal shape near to Clausen's obtained for compressed column only.

In Table 3. we present the minimum half volume $V_{\min}$, the minimal and maximal area of the bending rigidity for optimal column obtained for a few values of P and W and for a few values of φ_p. The bending rigidity of the uniform column of the same size as optimal EI_k and the corresponding value of the twisting angle at trivial equilibrium configuration φ_k are also presented.

Table 3.

W	P	φ_p	$V_{\min}$	$EI\,(0)$	$EI\,(1/2)$	EI_k	φ_k
0.05	9.868	0.073	0.436	0.071	1.315	0.758	0.043
		0.049	0.446	0.282	1.246	0.798	0.041
0.25	9.854	0.244	0.446	0.284	1.246	0.798	0.204
		0.179	0.480	0.750	1.085	0.921	0.177
1.0	9.620	0.878	0.454	0.411	1.205	0.823	0.790
		0.748	0.472	0.658	1.117	0.892	0.728

The values presented in Table 3 show that the savings in material, as one minus doubled value $V_{\min}$, are proportional to the difference of both $\varphi_p - \varphi_k$ and $EI\,(1/2) - EI\,(0)$. The last difference could represent a measure of difference between the optimal and uniform (cylindrical) column. The optimal shapes that correspond to the numerical solutions of the column are presented in Figure 5. As in previous section instead of $EI\,(S)$ we present optimal curve obtained as $R_{opt}\,(t) = EI\,(S)^{1/4}$, which according to Pearson formulation of the Lagrange problem, gives the compressed and twisted column of greatest efficiency.

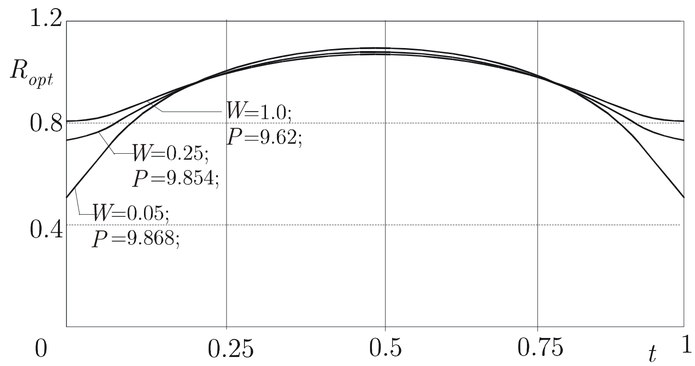


Figure 5. Optimal compressed and twisted columns against spatial buckling.

For comparison we note that the columns presented in Figure 5. could still remain straight but the uniform columns made of the same amount of material, loaded with the same P and W are very far away in the postcritical region. In order to illustrate this we shall examine the postbuckling behavior of the column of uniform cross section $EI_k = 0.823$, loaded with $P = 9.62$ and $W = 1.0$. For these values the corresponding nonlinear two point boundary value problem was solved numerically. In Figure 6. we present the projection of the column axis C on the Oxy plane. Note that the maximal moment corresponding to the trivial equilibrium configuration equals 1 and equals 3.305 in the buckled state. The corresponding twisting angle at the middle reads 0.378.

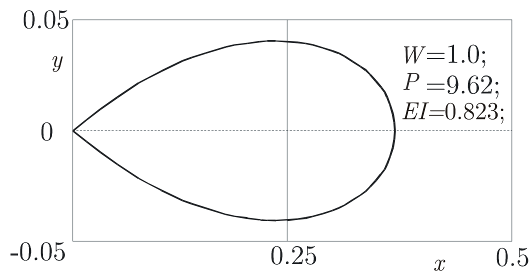


Figure 6. Postbuckling behavior of compressed and twisted uniform (cylindrical) column.

Similar proportions are valid for the other columns presented in Table 3. For example, for uniform column that correspond to the optimal one presented in the first raw the maximal moment in nontrivial equilibrium configurations reads 3.616. This value is in agreement with the one obtained for column presented in Fig. 3b equals 3.65, contributing to expectations that follow from the presented generalization of the Lagrange's problem.

Finally, it should be noted that the optimal shape problem for a compressed and twisted rod, according to authors's knowledge is completely new, and Table 3 as well as Figure 5 represent its first original solution.

In author's opinion further investigations related to this problem could be continued in series of different directions. For example, we could examine the problem for different type of boundary conditions. Besides we could use some other formulation of the optimal control problem. Along the lines of the remarks of the previous section we note that instead of Kirchhoff's model of elastic rod we could use some other. Generalized spatial elastica with shear and axial strain or constraints imposed on the bending rigidity deserves to be mentioned. On the other side we could examine optimal shapes of the columns for different type of the cross-section. At the end we believe that the optimal only twisted columns could be of some interest too.

4 A note on Kirchhoff's analogy

4.1 Preliminaries

In this section an application of analytical mechanics approach to the stability problem for compressed and twisted rod as well as the Euler-Poisson equations on Lie algebras will be discussed.

In his famous Lecture notes on Mechanics where basic facts on theoretical mechanics and mathematical methods of the last century were exposed, Kirchhoff made an analogy between the equilibrium equations of a spatially deformed rod and the equations describing rotation of a rigid body about a fixed point. Three questions concerning this analogy are considered here. The first one inquires the structure of the system of nonlinear differential equilibrium equations (1.7.6). In the second, the derivation of these equations from the fundamental principles of analytical mechanics is examined, while in the third, the possibility of casting the bifurcation of a trivial solution problem into an algebraic form is analyzed. As in the section on optimal shape of the compressed and twisted rod the classical theory without shear and compressibility is considered.

The leading fact in posing these questions was motivated by the work of Beljaev (1988) in which the analytical properties of the Euler-Poisson equations were given in the form

$$\begin{aligned} A_\xi \dot{\omega}_\xi &= (A_\eta - A_\zeta)\,\omega_\eta \omega_\zeta + r_\zeta v_\eta - r_\eta v_\zeta, \\ \dot{v}_\xi &= \omega_\zeta v_\eta - \omega_\eta v_\zeta, \end{aligned} \qquad (\xi,\eta,\zeta) \tag{3.1.1}$$

where dot denotes the derivative with respect to time, where $A_j > 0$, r_j represent real parameters $(j = \xi, \eta, \zeta)$ and where notation (ξ, η, ζ) denotes cyclic permutation of indices. In the case when $r_\xi = r_\eta = 0$, $r_\zeta = 1$, $A_\xi = a$, $A_\eta = b$, $A_\zeta = c$, $v_{(\cdot)} = V_{(\cdot)}$ (3.1.1) and the equilibrium equations of compressed and twisted rod (1.7.6) with $(e, f, g \to \infty)$ coincide. For the case of twisted rod the system (3.1.1) with $r_j = 0$, $(j = \xi, \eta, \zeta)$ reduces to the equilibrium equations of twisted, but not compressed, rod what was the special case of equilibrium equations analyzed in Grammel (1923) where v_j $(j = \xi, \eta, \zeta)$ correspond to the projections of the unit vector $\mathbf{k}$ on the axes of the moving coordinate system.

The identification of these systems in only formal because in (1.7.6), as well as in Section 2 of Grammel's work mentioned, dot denotes the derivative with respect to the arc length S, $S \in [0, 1]$. Thus, the application of analytical mechanics methods which are basically concerned with motion to problems connected to equilibrium of an elastic body should be done carefully.

The equations (3.1.1) are well known differential equations of a heavy rigid body rotating about a fixed point. The structure of these equations and the properties that follows from that structure are research subject of many authors, see Arnold (1969), Ratiu (1982), Ziglin (1983), Marsden *et al.* (1983). In the work of Beljaev as well as in other works mentioned, (3.1.1) were written in the Hamiltonian form on the orbits of certain products of the Lie algebras. The energy integral was used as the Hamiltonian. Although the questions on recognizing the Hamiltonian structure of elasticity were answered, see Simo *et al.* (1988), to made subtle connection between the main problem of this thesis and powerful methods of analytical mechanics,another approach will be adopted here. Namely, the equilibrium equations for twisted rod will be derived from the scalar energy functions by use of the Nambu mechanics developed in seventies.

4.2 Derivation of the equilibrium equations for twisted rod and the Nambu generalization of Hamiltonian dynamics

Motivated by the Liouville theorem, which states that the volume of phase space occupied by an ensemble of systems is conserved, and by the form of Euler's equations for a classical rigid rotator, Nambu (1973) suggested very interesting generalizations of classical Hamiltonian dynamics. Namely, introducing two functions

$$
\begin{aligned}
T_1 &= \frac{1}{2}\left(a\omega_\xi^2 + b\omega_\eta^2 + c\omega_\zeta^2\right), \\
T_2 &= \frac{1}{2}\left(\frac{a}{bc}\omega_\xi^2 + \frac{b}{ac}\omega_\eta^2 + \frac{c}{ab}\omega_\zeta^2\right),
\end{aligned}
\tag{3.2.1}
$$

which serve as a pair of "Hamiltonians" the differential equilibrium equations of a twisted rod, as postulated by Nambu for the case of the motion of a rigid body about a fixed point, could be obtained as

$$
\frac{d\boldsymbol{\omega}}{dS} = \bigtriangledown T_1 \times \bigtriangledown T_2, \tag{3.2.2}
$$

or in the developed form, see Nambu (1973),

$$
\dot{\omega}_\xi = \frac{\partial\left(T_1, T_2\right)}{\partial\left(\omega_\xi, \omega_\eta\right)}, \qquad (\xi, \eta, \zeta), \tag{3.2.3}
$$

where the use of partial derivatives notation on the right hand side pretends to generalize the Hamiltonian formalism. Note that in (3.2.3) dot represents the derivative with respect to S and (ξ, η, ζ) denotes cyclic permutation of indices, as before. Also, note that both T_1 and T_2 represent the first integrals of the obtained differential equations which coincide with $(1.7.6)_{4,5,6}$ for the case $V_j = 0, (j = \xi, \eta, \zeta)$.

The six differential equations representing equilibrium of a compressed and twisted rod (1.7.6) for the classical case $(e,\, f, g \to \infty)$, could be derived in sense of Nambu mechanics in form similar to (3.2.3), if one can find five energy functions of the kind similar to (3.2.1). The question of physical meaning of the needed functions in the rod theory is the question for itself and will not be treated here.

The equations in form (3.2.3) for the case of the Euler differential equations of a rigid rotator with the derivatives with respect to time, in tensor notation and a similar form of T_2 (in the form of the Casimir invariant, see Kentwell (1986)), could be found in the original Nambu's paper as well as in Bakai and Stepanovski (1981). The connection of the Nambu formulation and standard concepts of analytical mechanics attract attention of many authors, see Ruggieri (1976), Mukunda and Sudarshan (1976), Steeb and vanTonder (1988), Steeb and N. Euler (1988, 1991). Among their results, for the problem considered in this thesis, the most interesting is the one concerning the generalization of the Poisson brackets. Namely, in such a case the critical load could be determined from the complete system of first integrals i.e., from the algebraic equations. That question is considered in the next subsection.

4.3 A note on the possibility of solving the Grammel problem by alternative method

As a motivation for posing the problem let us consider the complete system of the first integral of (3.1.1) which was given in the form of the generalized Lie series in Lemains (1965, p. 133). In the implicit form these integrals could be written as

$$\begin{aligned} \omega_\xi &= \omega_\xi \left(S, \omega_\xi, \omega_\eta, \omega_\zeta, V_\xi, V_\eta, V_\zeta, \lambda, k_1, k_2, k_3, k_4, k_5, k_6\right), \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\xi, \eta, \zeta) \\ V_\xi &= V_\xi \left(S, \omega_\xi, \omega_\eta, \omega_\zeta, V_\xi, V_\eta, V_\zeta, \lambda, k_1, k_2, k_3, k_4, k_5, k_6\right). \end{aligned} \tag{3.3.1}$$

Supposing the Lie series involved in (3.3.1) are convergent on $[0, 1]$ we pose the question: how the critical buckling load could be determined from (3.3.1)? To answer the question one need seven conditions to determine the constants k_{1-6} and the load parameter λ. One possible answer is as follows. The constants k_{1-6} could be determined by use of (1.7.7) and λ could be determined from the definition of the bifurcation point given in Djukic and Atanackovic (1993, p.58). Namely, the critical load λ is determined when the norm of the difference of the solution (3.3.1) and their values corresponding to the trivial configuration vanishes.

Since (3.3.1) are rather complicated, there will be no numerical experiments in this thesis. The complexity of (3.3.1) increases the need for the independent first integrals of (3.1.1) what could be seen in almost all the papers cited in the previous subsection. Besides Bakai and Stepanovski (1981) very comprehensive list of references concerning first integrals is given in Vujanovic and Jones (1989).

5 The loss of stability of a compressed and twisted rod

5.1 Preliminaries

In this section spatial elastica theories with shear and axial strain are considered again. If the dynamical concepts of the temporal and structural stability are followed, see Dole and Norbury (1991, p. 4), then it should be noted that in Subsection 1.8 the first concept was examined. Namely, if the rod is loaded with $\lambda << \lambda_{cr}$ no matter how large disturbances in $\psi(0)$ and $\vartheta(0)$ are, the only solution of the corresponding equilibrium equations is the one which corresponds to the trivial configuration. In the case when λ approaches a neighborhood of λ_{cr} for some specific values of $\psi(0)$ and $\vartheta(0)$ the solution set could become different.

The second concept, the structural stability, is concerned with the influence of perturbations of the differential equilibrium equations on the solution set. In general both problems belong to the theory presented in Golubitsky and Shaeffer (1985). In this part the recognition problem i.e. the bifurcation pattern for λ in the neighborhood of λ_{cr} will be solved. Since the singularity theory deals with algebraic equations in the next section the Liapunov-Schmidt reduction procedure will be performed. Namely, the two-point boundary value problem corresponding to the equilibrium equations of the compressed and twisted rod, will be transformed to the equation

$$m^3 \mp m \triangle \lambda = 0, \tag{4.1.1}$$

where $m \in \mathbb{R}$, $(m \to 0)$, and $\triangle\lambda$ is the small change in the bifurcation (load) parameter. Only the perfect case will be analyzed i.e., the assumptions introduced in Subsections 1.4-7 are again under consideration. The next step in applying singularity theory could be the universal unfolding problem. connected with (4.1.1). Thus, the imperfect bifurcation of the rod problem will be discussed involving changes of physical assumptions on the load and form in undeformed state of the rod. If the imperfections are to be introduced the differential equilibrium equations could differ from those presented in Subsection 1.7. To questions what is the form of these equations in such a case and how the new parameters describing imperfections change the bifurcation pattern corresponding to the perfect case the answers will not be considered. The main reasons for avoiding these answers are the fact that the physical description of the spatially deformed geometrically imperfect rod in the literature on the subject is not stationary jet and the high dimension of the problem.

5.2 Liapunov-Schmidt reduction procedure and bifurcation patterns for a perfect rod

The preparation needed was almost done in Subsection 1.8 so the additional result to Theorem 1 follows

Theorem 3 *For small enough values of $\triangle\lambda$ all solutions of (1.8.3) which are for $\lambda =$*

$\lambda_{cr} + \triangle\lambda$ in the neighborhood of $0 \in H_0^1$ could be represented in the form

$$u = mu_s + u^*, \tag{4.2.1}$$

where u_s represents the eigenfunction of the linearized problem (1.8.6) which corresponds to λ_{cr}, m is the real number that satisfies the bifurcation equation

$$c_3 m^3 + c_1 m \triangle \lambda + h.o.t. = 0, \tag{4.2.2}$$

where $h.o.t.$ denotes higher order terms $O\left(m^4, m^3 \triangle \lambda\right)$. Further u^* is the continuous function of order m^3 which satisfies the condition

$$\int_0^1 \mathcal{DF} u^* q_s dS = 0, \tag{4.2.3}$$

where q_s represents eigenvector of the adjoint problem that corresponds to λ_{cr}. Finally, c_1 and c_3 are the constants depending on λ_{cr}, e, f, g, a, b and c.

Proof. The relation (4.2.1) is the consequence of the fact that the eigenvalues of the linearized problem are geometrically simple. The remaining assessments will be proved by use of the Liapunov-Schmidt reduction. Following the standard references Chow & Hale (1981) and Golubitsky & Schaeffer (1985), we conclude that each $u \in H_0^1$ (as well as the solution to (1.8.3)) could be decomposed as shown in (4.2.1). Now, u given by (4.2.1) is the solution of (1.8.3) in the neighborhood of zero, if and only if, the parameter m satisfies the following bifurcation equation

$$G(\lambda, u) = \int_0^1 \left[\mathcal{L}\left(\lambda_{cr} + \triangle\lambda\right)\left(mu_s + u^*\right) + \mathcal{N}\left(\left(\lambda_{cr} + \triangle\lambda\right), \left(mu_s + u^*\right)\right)\right]^{\mathrm{T}} q_s dS = 0. \tag{4.2.4}$$

Since u_s satisfies (1.8.6), and since the vectors u^* and q_s are orthogonal, in the case of expansion to third order terms, the equation (4.2.3) reduces to (4.2.2) with coefficients c_1

and c_3 (obtained after lengthy calculations) as

$$c_1 = CD \int_0^1 \left[\left(\tfrac{1}{b} - \tfrac{1}{c}\right) u_2 q_1 + \left(\tfrac{1}{c} - \tfrac{1}{a}\right) u_2 q_1 + \tfrac{1}{c} \left(u_4 q_5 - u_5 q_4\right) \right] dS,$$

$$\begin{aligned} c_3 = C^3 D \int_0^1 \Bigg\{ & \left[\frac{\lambda_{cr}}{2} \left(\frac{1}{c} - \frac{1}{b} \right) q_1 + \frac{1}{2} \left(\frac{1}{f} - \frac{1}{g} \right) q_4 \right] \left(u_1^2 u_2 + u_2^3 \right) - \\ & \left[\frac{\lambda_{cr}}{2} \left(\frac{1}{c} - \frac{1}{a} \right) q_2 + \frac{1}{2} \left(\frac{1}{e} - \frac{1}{g} \right) q_5 \right] \left(u_1^3 + u_1 u_2^2 \right) + \\ & \left[\frac{q_1}{2b} + \frac{\lambda_{cr} q_4}{2c} \right] \left(u_1^2 + u_2^2 \right) u_5 - \left[\frac{q_2}{2a} + \frac{\lambda_{cr} q_5}{2c} \right] \left(u_1^2 + u_2^2 \right) u_4 - \\ & \left[\frac{q_1}{c} + \frac{\lambda_{cr} q_4}{b} \right] \left(u_1 u_2 u_4 + u_2^2 u_5 \right) + \\ & \left[\frac{q_2}{c} + \frac{\lambda_{cr} q_5}{a} \right] \left(u_1 u_2 u_5 + u_1^2 u_4 \right) - \\ & \left(\frac{1}{c} - \frac{1}{b} \right) \left(u_1 u_4 u_5 + u_2 u_5^2 \right) q_4 + \\ & \left(\frac{1}{c} - \frac{1}{a} \right) \left(u_2 u_4 u_5 + u_1 u_4^2 \right) q_5 \Bigg\} dS, \end{aligned} \tag{4.2.5}$$

where u_i and q_i, $(i = 1, 2, 4, 5)$ are given by the series (1.8.7) and (1.8.12) respectively. The function u^* is at most $O\left(m^2\right)$, and because of (1.8.2) $O\left(m^3\right)$ what follows from the implicit function theorem. The proof is complete.

In order to determine the number of solutions of (4.2.2) the coefficients c_1 and c_3 will be calculated. In the case when c_1 and c_3 differ form zero the problem (4.2.2) is contact equivalent to (4.1.1), see Keyfitz (1986). The bifurcation diagram is the familiar pitchfork. When c_1 and c_3 are of different sign, in the space considered here, the bifurcation is supercritical. When the signs of c_1 and c_3 are the same the rod exhibits subcritical bifurcation. If either c_1 or c_3 vanish for analysis of the bifurcation diagrams higher order terms should be included in (4.2.2) and (4.2.4). The special case of that kind will not be considered here. To illustrate the previous analysis, by use of the values presented in Table 1, the values of the coefficients c_1 and c_3 are calculated and given in Table 3.

Table 3. The coefficients c_1 and c_3 and the corresponding bifurcation type.

Rod description $b = (\kappa\pi^2)^{-1}, \quad a = 2b = 3c$					Pitchfork $c_3 m^3 + c_1 m \triangle \lambda = 0$		
κ	ρ	e	f	g	c_1	c_2	bifurcation type
2/3	100	$\to\infty$	$\to\infty$	$\to\infty$	0.59566	0.50095	subcritical
		$\to\infty$	$\to\infty$	1520	0.59632	0.50202	subcritical
		456	304	$\to\infty$	0.59236	0.49667	subcritical
		456	304	1520	0.59270	0.49773	subcritical
	10	$\to\infty$	$\to\infty$	15.2	1.10524	0.75914	subcritical
		4.56	3.04	$\to\infty$	-1.58935	3.00456	supercritical
		4.56	3.04	15.2	-2.44637	12.80152	supercritical
19/20	100	$\to\infty$	$\to\infty$	$\to\infty$	-0.81306	0.24338	supercritical
		$\to\infty$	$\to\infty$	1066	-0.81439	0.24447	supercritical
		320	213	$\to\infty$	-0.81048	0.24242	supercritical
		320	213	1066	-0.81228	0.24283	supercritical
	10	$\to\infty$	$\to\infty$	66.7	-0.84571	0.25915	supercritical
		20	13.3	$\to\infty$	-0.79460	0.65939	supercritical
		20	13.3	66.7	-0.81341	0.29403	supercritical

From the results presented in Table 3 we may conclude that for certain values of the load parameter κ and for specified values of the rod parameters ρ, e, f, g the different types of bifurcation may occur. The fact that κ could change the type of the bifurcation in generalized spatial elastica theory, represents generalization of the result obtained by Beda (1990), which refers to the classical Kirchhoff theory. The fact that the parameters describing the influence of shear and compressibility for the fixed load parameter, as shown, also could change the type of bifurcation, represents generalization of the plane elastica theory, see Atanackovic (1989), Atanackovic and Djukic (1992) and Atanackovic and Spasic (1992).

5.3 On imperfections in load

In this and in the next subsection some possibilities of further research concerned with the stability problem of a compressed and twisted rod with shear and axial strain will be discussed.

So far the rod with an end load was considered, and the rod weight was neglected. This means the weight of the rod was much smaller than the compressing force. First, the heavy rod in either vertical or horizontal position could be considered as a possible generalization. That kind of problem requests introducing the weight of rod of unit length in an undeformed state, say $\mathbf{q}_0$. Since, the mass of the rod in the deformed and undeformed state is the same, by use of (1.3.1) we should take $\mathbf{q} = \mathbf{q}_0/(1+\varepsilon)$ into static equations of Subsection 1.4. If $|\mathbf{q}|$ is taken to be small enough the problem that follows could be treated as the imperfection of the basic rod problem considered here. In such a case all nonlinear and linearized equations considered after Subsection 1.4 will become more complicated.

Almost the same happens if one introduce the distributed couples along the rod axis. In that case $\mathbf{m} \neq \mathbf{0}$ should be put in the equations of Subsection 1.4. Physically that corresponds to a rod in a turbulent fluid stream. Along that lines, the stability of the vertically positioned immersed rod under hydrostatic pressure, for which $\mathbf{q} = \mathbf{q}(S)$, could be considered. For example see Tucsnak and Sebe (1989).

Finally, once again we mention the work of Atanackovic (1989), in which $\mathbf{q}$ is taken in the form that gives a possibility to model a small concentrated force acting at the middle point of the rod span. This type of imperfection seems to be very interesting because of applications in industrial and engineering fields.

5.4 On imperfections in shape

This subsection deals with the problem how a rod which is not straight and prismatic in the undeformed state could be described for the purpose of stability analysis. In the classical case, the imperfect rod was considered by Kirchhoff 1859 and by Clebch in 1862 . The same constitutive relations, but in different manner, were obtained by Basset in 1895. Besides Love (1927, p. 397) the geometrically linear version of the Kirchhoff-Clebch theory could be found in Filin *et al.* (1981). The analysis of the geometrically imperfect rods is more complicated even in the simple case of only compressed (nontwisted) rod. Namely, the compressed rod which is in the undeformed state straight and prismatic, deforms in plane, what is not the case if the rod axis is curved and twisted in the undeformed state. In such a case the spatial deformation occurs.

In the case of spatially deformed rods with shear and axial strain the things use to be more complicated, and thus there is no comprehensive theory which will take all recognized effects into account, see Filin *et al.* (1981). Despite the fact that in the monograph of Filin *et al.* the shear and compressibility were considered the constitutive equations were given only for the case of straight and prismatic rod. The question of such relations is still open as could be seen in Rosen (1991), where experimental results cited emphasize the fact that changes of the cross-section should be included in theories of the imperfect rods. On the other hand plane elastica theories, and the elasticity theory, prefer that the coupling effects should be taken into account. Namely, the bending-twisting and extension-twisting couplings as well as their integral contribution to the resultant force and couple in an arbitrary cross-section are to be included within constitutive equations. The answer how the coupling effects should be analyzed is not unique even in the plane theories. For example, in the case of generalizing (1.1.2) the imperfect rod is described as

$$
M = -E \left(\int_A \frac{n^2}{1 + \dfrac{n}{R}} dA \right) \left(\dot{\vartheta} - \dot{\gamma} + \frac{1}{R} \right),
$$

$$
N_H = E \left(\int_A \frac{dA}{1 + \dfrac{n}{R}} \right) \left[(1 + \varepsilon) \cos \gamma - 1 \right], \tag{4.4.1}
$$

$$
Q_H = \frac{G}{k} \left(\int_A \frac{dA}{1 + \dfrac{n}{R}} \right) (1 + \varepsilon) \sin \gamma,
$$

where n measures the distance of the rod fiber from the rod axis and R is the radius of curvature of the rod axis in the unloaded state, see Goto *et al.* (1990), while in the case of generalizing (1.1.4) with the adopted notation the rod model is

$$
M = -E(\dot{\vartheta} - \dot{\gamma}) \left(\int_A \frac{n^2}{1 + \dfrac{n}{R}} dA \right) - E\varepsilon \left(\int_A \frac{n}{1 + \dfrac{n}{R}} dA \right),
$$

$$
N_E = E\varepsilon \left(\int_A \frac{dA}{1 + \dfrac{n}{R}} \right) + E \left(\dot{\vartheta} - \dot{\gamma} \right) \left(\int_A \frac{n}{1 + \dfrac{n}{R}} dA \right), \tag{4.4.2}
$$

with $(1.1.4)_3$ in the same form, see Atanackovic and Spasic (1992). It is not necessary to emphasize that there are some other solutions.

The deformation coupling problem occupies the central place in spatial theories as well. The relations (1.5.1) used here, for the case of imperfect rod are generalized in the following form

$$
\begin{aligned}
\mathbf{M} &= \mathbf{A}\mathbf{W}_1 + \mathbf{C}\mathbf{T}, \\
\mathbf{V} &= \mathbf{B}\mathbf{T} + \mathbf{C}\mathbf{W}_1,
\end{aligned} \tag{4.4.3}
$$

where $\mathbf{C}$ is the nonsymmetric stiffness tensor that connects elastic properties of the rod material with the shape and dimensions of the cross-section, and involves the coupling effects, while $\mathbf{W}_1 = \mathbf{W} - \mathbb{P}\mathbf{W}_0$ where $\mathbf{W}_0$ represents the curvature and torsion of the rod in the undeformed state, see Eliseyev (1988). The form of $\mathbf{C}$ is a problem for itself (except in the case of the perfect rod when $\mathbf{C} = \mathbf{0}$), and is not an easy one, see Berdichevskii and Staroselski (1983,1988), Magomaev (1985), Eliseyev (1988) and Krenk (1983), for example. The constitutive relations presented in Simo *et al.* (1988) do not consider the coupling problem but in some way generalize the well known Kirchhoff-Clebch theory presented in Love (1923). Namely, the rod model in Simo *et al.* (1988) is given in the form

$$\mathbf{M} = \mathbf{A}\mathbf{W}_1,$$
$$\mathbf{V} = \mathbf{B}\mathbf{T}_1, \tag{4.4.4}$$

with $\mathbf{T}_1 = \mathbf{T} - \mathbb{P}\mathbf{T}_0$, where $\mathbf{T}_0$, defined in the same manner as $\mathbf{T}$ of Section 1.5, was used to describe the undeformed imperfect configuration. Both (4.4.3) and (4.4.4) are not easy to apply in technical applications, especially in the stability analysis.

For the purpose of examining the stability of a spatially deformed imperfect rod the relations presented in Kingsbury seem to be more tractable whose approach is a little bit different then the one that was followed here. The constitutive relations of Kingsbury are written in the natural system (the tangent, principal normal and bi-normal) which is, in general, inclined with respect to principal directions of the cross-section area. The resultant force and the resultant couple were presented as functions of the displacement of the point on the rod axis with respect to the undeformed state and small angles describing rotation of the cross-section about the natural axes. The explicit form of the Kingsbury relations will not be written here but it should be noted that his work presents the different type of decomposition of the resultant force (Timoshenko's approach) what, by predictions of the plane elastica theories, see Gjelsvik (1991) and Atanackovic and Spasic (1992), could result in the critical load different from the values presented Table 1 where the Haringx approach was followed.

The completely different rod models then (4.4.3) and (4.4.4) could be found in Goto *et al.* (1985) and Iura and Hirashima (1992).

We end this section by noting that beside the question which rod model should be chosen among the existing relations of Haringx's and Timoshenko's approach, the question of developing constitutive theory for Engesser's approach in spatial case, for the purpose of further research, could also be posed.

6 Conclusions

In this thesis the stability problem of a compressed and twisted rod was examined. The load and the supports corresponding to nonconservative model were adopted. The constitutive relations of the generalized elastica theory as to take shear and compressibility into account,were taken in the form proposed by Eliseyev (1988) with Haringx's type of decomposition of the resultant elastic force in an arbitrary cross-section. The finite deformations were analyzed.

On the basis of the nonlinear equilibrium equations derived (Subsection 1.7) the bifurcation analysis was performed, (Subsections 1.8 and 4.2). It was shown that the bifurcation points of linearized equations determine the bifurcation points of nonlinear equilibrium equations as well as that the bifurcation pattern corresponds to familiar pitchfork. For specified values of the rod parameters the critical load, the postcritical shape and the bifurcation type were determined numerically, (Subsections 1.9, 5.2 and 4.2). It could be noted that the obtained results generalize plane elastica theories with shear and compressibility in a natural way. Also, some results generalize the classical Kirchhoff theory. The dependance of the critical load parameter for a compressed and twisted rod on the parameters describing shear and compressibility, presented in Table 1, represents the principal novelty of this work.

The nonlinear analysis presented was used in the formulation of the problem considering optimal shape of the compressed and twisted rod against buckling. The problem was posed for specified values of the rod parameters which correspond to the classical theory which does not take effects of shear and compressibility into account. It seems to be regular since those effects have the opposite influence on the buckling load. By use of the Pontriagin maximum principle the necessary conditions for extremum were derived. For specially chosen formulation the optimal shape of the rod and the prediction of its efficiency compared to the (usually built) cylindrical rod were determined numerically. The results presented are completely new as far as author's knowledge was concerned.

In this work some attempts were made in application of analytical mechanics methods to several segments of the rod problem. Hence, for the case of the classical theory for a twisted column of arbitrary cross-section the differential equilibrium equations were derived by use of the scalar energy functions and the Nambu formulation of Hamiltonian dynamics (Subsection 3.2). The problem of algebraic bifurcation that corresponds to the stability problem was also discussed (Subsection 3.3).

Some further directions of possible research on the stability problem of a compressed and twisted elastic rod were given (Subsections 4.3 and 4.4). Namely, the problem considered here could be generalized as to take some other constitutive models as well as the different load. Also, further investigations of the optimal control problem for both plane and spatial stability problems are presented in Section 2.

Finally, the reference list given at the end may be treated as a result of this research too because it contains very different and in author's opinion, only at first sight, unconnected areas of the mathematical theory of elastic rods.

7 Appendix

7.1 A note on the Krylov (ship) angels

Consider two right handed coordinate systems, one fixed $Oxyz$ and the other $O\xi\eta\zeta$ that could rotate about O in any direction. Any possible orientation of the second system could be described in terms of a rotation ψ about the y axis, then a rotation ϑ about the new x axis (i.e., the direction of x axis obtained in preceding rotation) and finally, a rotation φ about the new z axis. All rotations are performed counterclockwise. In the following figure the successive rotations from the top of the axis about which the system rotate are shown together with the immediate positions. The unit vectors of the system $Oxyz$, say **i**, **j** and **k**, and the system $O\xi\eta\zeta$, say **a**, **b** and **c** are also shown.

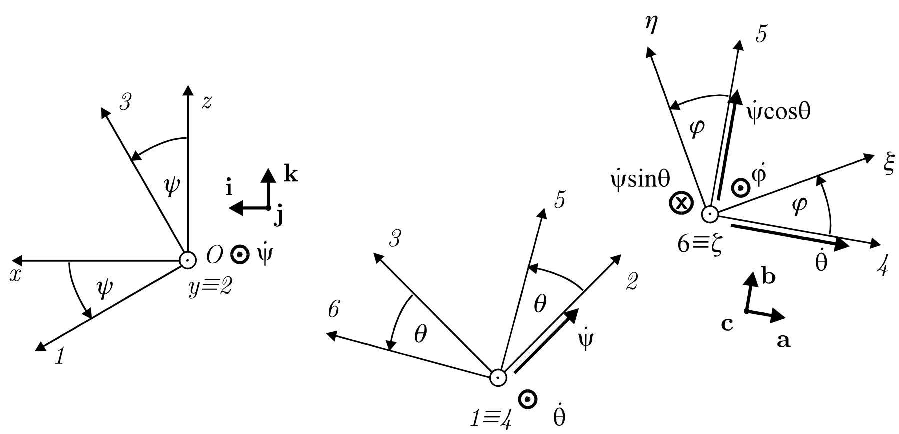


Fig. 3. The Krilov angles.

The motivation for choosing these angles is their property that the small difference in orientation of the systems $Oxyz$ and $O\xi\eta\zeta$ restricts the value of each of these angles, what is not the case when dealing with usual (continental) version of the Euler angles in which the nutation angle and the sum of the other two angles remain small. The last could be significant if the procedure of linearization is to be performed, see Lurie (1961, 52). Note that this type allows the first two angles to be small and the third one to be finite. In the Russian literature these angles are called ship or Krilov angles.

According to that sequence of rotations the following geometrical relations hold:

- the components of the angular velocity $\boldsymbol{\omega} = \dot{\boldsymbol{\psi}} + \boldsymbol{\vartheta} + \dot{\boldsymbol{\varphi}}$ in the system $O\xi\eta\zeta$ in terms of the rotation speeds (where dot denotes the derivative with respect to some parameter), see Fig. 3

$$\omega_\xi = \dot{\psi}\cos\theta\sin\varphi + \dot{\theta}\cos\varphi,$$

$$\omega_\eta = \dot{\psi}\cos\theta\cos\varphi - \dot{\theta}\sin\varphi, \tag{5.1.1}$$

$$\omega_\zeta = -\dot{\psi}\sin\theta + \dot{\varphi},$$

- the rotation speeds solved in terms of that components

$$\dot{\psi} = \frac{\omega_\xi \sin\varphi + \omega_\eta \cos\varphi}{\cos\vartheta},$$

$$\dot{\theta} = \omega_\xi \cos\varphi - \omega_\eta \sin\varphi, \tag{5.1.2}$$

$$\dot{\varphi} = \omega_\zeta + (\omega_\xi \sin\varphi + \omega_\eta \cos\varphi)\tan\vartheta,$$

- the relations between the unit vectors of the introduced systems

$$\mathbf{a} = (\cos\psi\cos\varphi + \sin\psi\sin\theta\sin\varphi)\,\mathbf{i} + \cos\vartheta\sin\varphi\;\mathbf{j} + \\ + (\cos\psi\sin\theta\sin\varphi - \sin\psi\cos\varphi)\,\mathbf{k},$$

$$\mathbf{b} = (\sin\psi\sin\theta\cos\varphi - \cos\psi\sin\varphi)\,\mathbf{i} + \cos\theta\cos\varphi\;\mathbf{j} + \\ (\sin\psi\sin\varphi + \cos\psi\sin\theta\cos\varphi)\,\mathbf{k}, \tag{5.1.3}$$

$$\mathbf{c} = \sin\psi\cos\theta\;\mathbf{i} - \sin\theta\;\mathbf{j} + \cos\psi\cos\theta\;\mathbf{k},$$

- the change of unit vectors

$$\dot{\mathbf{a}} = \omega_\zeta\;\mathbf{b} - \omega_\eta\;\mathbf{c},$$

$$\dot{\mathbf{b}} = \omega_\xi\;\mathbf{c} - \omega_\zeta\;\mathbf{a}, \tag{5.1.4}$$

$$\dot{\mathbf{c}} = \omega_\eta\;\mathbf{a} - \omega_\xi\;\mathbf{b},$$

- the rotation tensor $\mathbb{P} = \mathbf{a}\odot\mathbf{i} + \mathbf{b}\odot\mathbf{j} + \mathbf{c}\odot\mathbf{k}$, as one-parametric orthogonal transformation (where $\odot$ denotes the diad product of vectors)

$$\mathbb{P} = \begin{bmatrix} \cos\psi\cos\varphi + \sin\psi\sin\theta\sin\varphi & \sin\psi\sin\theta\cos\varphi - \cos\psi\sin\varphi & \sin\psi\cos\theta \\ \cos\vartheta\sin\varphi & \cos\theta\cos\varphi & -\sin\theta \\ \cos\psi\sin\theta\sin\varphi - \sin\psi\cos\varphi & \sin\psi\sin\varphi + \cos\psi\sin\theta\cos\varphi & \cos\psi\cos\theta \end{bmatrix}. \tag{5.1.5}$$

If the unit vector $\mathbf{c}$ is to be transformed to coincide with the unit vector of an arbitrary direction, say $\mathbf{t}$, two additional angles of the Euler type, say α and β, will be introduced in the same way as the angles ψ and ϑ above.

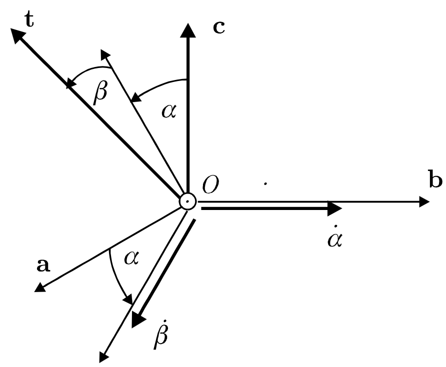


Fig. 4. The additional angles.

The following relation determines $\mathbf{t}$ in the coordinate system $O\xi\eta\zeta$ see Fig. 4

$$\mathbf{t} = \sin\alpha\cos\beta\ \mathbf{a} - \sin\beta\ \mathbf{b} + \cos\alpha\cos\beta\ \mathbf{c}. \tag{5.1.6}$$

7.2 Some results of the numerical experiments

In Table 2 of Subsection 1.9 only the maximal values of the components of solutions of (1.7.4) were presented. The following tables contain some more results of the numerical experiments performed as described in Section 1. 9.

Kirchhoff's model								
λ	κ	ρ	a	b	c	e	f	g
0.16	19/20	100	0.2133	0.1067	0.0711	$\to\infty$	$\to\infty$	$\to\infty$

S	x	y	z	ψ	θ	φ	α	β	ε	$\|\mathbf{M}\|$
0	0	0	0	-.2703	.2686	0	0	0	0	.16
.1	-.0235	-.0272	.0933	-.2185	.2778	.2237	0	0	0	.1640
.2	-.0411	-.0540	.1880	-.1471	.2608	.4534	0	0	0	.1738
.3	-.0515	-.0777	.2846	-.0664	.2131	.6837	0	0	0	.1852
.4	-.0541	-.0953	.3829	.0108	.1353	.9086	0	0	0	.1939
.5	-.0498	-.1039	.4824	.0728	.0348	1.1253	0	0	0	.1972
.6	-.0403	-.1019	.5819	.1124	-.0763	1.3360	0	0	0	.1939
.7	-.0283	-.0888	.6802	.1266	-.1841	1.5456	0	0	0	.1852
.8	-.0163	-.0659	.7768	.1168	-.2755	1.7592	0	0	0	.1738
.9	-.0065	-.0353	.8715	.0882	-.3409	1.9786	0	0	0	.1640
1	0	0	.9648	.0489	-.3757	2.2021	0	0	0	.16

Eliseyev's model								
λ	κ	ρ	a	b	c	e	f	g
0.16	19/20	100	0.2133	0.1067	0.0711	320	210	1100

S	x	y	z	ψ	θ	φ	α	β	ε	$\|\mathbf{M}\|$
0	0	0	0	-.2873	.2844	0	-.89	1.28	-.84	.16
.1	-.0248	-.0288	.0924	-.2325	.2942	.2235	-.9	1.07	-.85	.1645
.2	-.0435	-.0573	.1863	-.1566	.2763	.4539	-.81	.83	-.86	.1754
.3	-.0545	-.0825	.2823	-.0708	.2257	.6849	-.61	.61	-.88	.1881
.4	-.0574	-.1011	.3804	.0110	.1432	.9098	-.33	.46	-.9	.1978
.5	-.0528	-.1103	.4797	.0767	.0367	1.1256	0	.4	-.91	.2014
.6	-.0428	-.1081	.5790	.1187	-.0811	1.3346	.33	.46	-.9	.1978
.7	-.0301	-.0942	.6770	.1341	-.1952	1.5423	.61	.61	-.88	.1881
.8	-.0174	-.0699	.7731	.1240	-.2921	1.7544	.81	.83	-.86	.1754
.9	-.0069	-.0374	.8670	.0939	-.3614	1.9731	.9	1.07	-.85	.1645
1	0	0	.9594	.0521	-.3982	2.1964	.89	1.28	-.84	.16

(with α, β and ε multiplied by 10^{-3}).

Eliseyev's model								
λ	κ	ρ	a	b	c	e	f	g
0.16	19/20	25	0.2133	0.1067	0.0711	20	13.3	66.6

S	x	y	z	ψ	θ	φ	α	β	ε	$\lvert\mathbf{M}\rvert$
0	0	0	0	-.4600	.4346	0	-.0225	.0286	-.0115	.16
.1	-.0377	-.0451	.0794	-.3750	.4512	.2213	-.0226	.0240	-.0120	.1705
.2	-.0663	-.0898	.1626	-.2534	.4252	.4609	-.0204	.0187	-.0128	.1951
.3	-.0835	-.1295	.2512	-.1166	.3470	.7025	-.0156	.0139	-.0138	.2222
.4	-.0883	-.1589	.3450	.0105	.2180	.9290	-.0085	.0105	-.0146	.2422
.5	-.0815	-.1732	.4421	.1105	.0518	1.1337	0	.0093	-.0148	.2495
.6	-.0662	-.1693	.5391	.1755	-.1303	1.3223	.0085	.0105	-.0146	.2422
.7	-.0466	-.1469	.6330	.2031	-.3057	1.5073	.0156	.0139	-.0138	.2222
.8	-.0270	-.1084	.7215	.1930	-.4543	1.7011	.0204	.0187	-.0128	.1951
.9	-.0108	-.0578	.8047	.1493	-.5608	1.9103	.0226	.0240	-.0120	.1705
1	0	0	.8841	.0833	-.6170	2.1318	.0225	.0286	-.0115	.16

Kirchhoff's model								
λ	κ	ρ	a	b	c	e	f	g
0.708	2/3	100	0.304	0.152	0.1013	$\to\infty$	$\to\infty$	$\to\infty$

S	x	y	z	ψ	θ	φ	α	β	ε	$\lvert\mathbf{M}\rvert$
0	0	0	0	-.0272	1.067	0	0	0	0	.708
.1	.0186	-.0842	.0493	.6360	.8841	.9012	0	0	0	.7132
.2	.0718	-.1484	.1032	.8556	.5048	1.389	0	0	0	.7269
.3	.1443	-.1770	.1646	.8560	.0785	1.732	0	0	0	.7439
.4	.2145	-.1640	.2335	.7054	-.3346	2.092	0	0	0	.7577
.5	.2600	-.1156	.3071	.3491	-.6560	2.604	0	0	0	.7630
.6	.2655	-.0493	.3807	-.2277	-.7404	3.320	0	0	0	.7577
.7	.2281	.0115	.4495	-.7110	-.5308	3.952	0	0	0	.7439
.8	.1582	.0462	.5109	-.9495	-.1671	4.378	0	0	0	.7269
.9	.0750	.0426	.5648	-1.018	.2411	4.720	0	0	0	.7132
1	0	0	.6141	-.9258	.6393	5.103	0	0	0	.708

Eliseyev's model								
λ	κ	ρ	a	b	c	e	f	g
0.708	2/3	100	0.304	0.152	0.1013	460	300	1500

S	x	y	z	ψ	θ	φ	α	β	ε	$\lvert\mathbf{M}\rvert$
0	0	0	0	-.0300	1.070	0	-.07	2.92	-.32	.708
.1	.0186	-.0845	.0487	.6374	.8869	.9040	-.26	2.84	-.33	.7133
.2	.0718	-.1489	.1023	.8571	.5068	1.391	-.39	2.67	-.38	.7270
.3	.1445	-.1777	.1633	.8573	.0800	1.732	-.38	2.46	-.43	.7441
.4	.2149	-.1648	.2318	.7067	-.3338	2.091	-.23	2.29	-.48	.7580
.5	.2606	-.1163	.3052	.3506	-.6561	2.602	0	2.23	-.49	.7634
.6	.2662	-.0499	.3786	-.2270	-.7415	3.317	.23	2.29	-.48	.7580
.7	.2288	.0112	.4471	-.7119	-.5323	3.950	.38	2.46	-.43	.7441
.8	.1588	.0460	.5081	-.9514	-.1684	4.376	.39	2.67	-.38	.7270
.9	.0753	.0426	.5617	-1.021	.2402	4.716	.26	2.84	-.33	.7133
1	0	0	.6104	-.9301	.6394	5.097	.07	2.92	-.32	.708

(with α, β and ε multiplied by 10^{-3}).

Eliseyev's model								
λ	κ	ρ	a	b	c	e	f	g
0.708	2/3	10	0.304	0.152	0.1013	4.56	3.04	15.2

S	x	y	z	ψ	θ	φ	α	β	ε	$\lvert\mathbf{M}\rvert$
0	0	0	0	-.1783	1.213	0	-.0398	.3006	.0240	.708
.1	.0185	-.0991	.0084	.7115	1.005	1.069	-.0554	.2887	.0169	.7151
.2	.0785	-.1771	.0271	.9165	.5908	1.498	-.0635	.2704	.0052	.7340
.3	.1636	-.2151	.0596	.8956	.1414	1.782	-.0570	.2497	-.0084	.7578
.4	.2486	-.2036	.1057	.7344	-.2944	2.094	-.0341	.2330	-.0197	.7775
.5	.3054	-.1480	.1604	.3718	-.6431	2.565	0	.2266	-.024	.7851
.6	.3138	-.0691	.2151	-.2340	-.7522	3.265	.0341	.2330	-.0197	.7775
.7	.2702	.0047	.2613	-.7610	-.5462	3.890	.0570	.2497	-.0084	.7578
.8	.1876	.0480	.2938	-1.032	-.1733	4.279	.0635	.2704	.0052	.7340
.9	.0893	.0468	.3124	-1.138	.2505	4.553	.0554	.2887	.0169	.7151
1	0	0	.3208	-1.111	.6801	4.824	.0398	.3006	.0240	.708

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