Optimal design of elastic columns for maximum buckling load

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Abstract

The problem of Lagrange, to find the curve which by its revolution about an axis in its plane determines the column of greatest efficiency, is examined. A comparison is made between the optimal shapes of the compressed column predicted by several existing formulations for columns of circular cross section hinged at the end points. Then, two different generalizations of the problem, that follow from a generalized plane elastica and the theory called Kirchhoff's kinetic analogue, are considered. The optimal shape of a compressed column that can suffer not only flexure as in classical elastica theory, but also compression and shear is first presented. Second, the distribution of material along the length of a compressed and twisted column is optimized so that the column is of minimum volume and will support a given load without spatial buckling. Necessary conditions for both problems are derived using the maximum principle of Pontryagin. The optimal shapes are obtained by numerical integration. The principal novelty of the present results is that both solutions, that follow from two possible generalizations of the classical Bernoulli-Euler bending theory, lead to the optimum column with non-zero cross sectional area at its ends.

1 Introduction

The problem of determining that shape of compressed column which has the largest Euler buckling load was posed by Lagrange in 1773. Clausen in 1851 solved it for columns of circular cross section pined at the end points. Although that result was mathematically correct the obtained optimal shape did have points where the cross section vanishes. Nikolai [1], in work long unnoticed outside the Soviet Union, was the first author who considered that anomaly of Clausen's solution. In order to avoid any finite load to induce infinite stresses in the column, Nikolai proposed minimal cross sectional area at the ends, determined so that given limiting stress will not be exceeded. Since then, many results of structural optimization could be related to the problem of Lagrange. Mathematically, this amounts to maximizing an eigenvalue of a certain Sturm-Liouville system to obtain an isoperimetric inequality. It is worth noting that the problem of Lagrange with clamped-clamped boundary conditions, was attacked in 1962 by Tadjbakhsh and Keller [2] in the continuation of work Keller [3] had begun at the suggestion of Clifford Truesdell. That case, in which the column is clamped at each end, seems to be especially troublesome since the solution has two points in its interior where the cross sections vanishes.

It is obvious that these extremal shapes have led to confusion and several attempts to resolve the anomaly. Nearly in the same manner as Nikolai, Trahair and Booker [4] have considered the problem of avoiding the zero cross sectional area at the end of the optimal column. In 1977 Olhoff and Rasmussen [5] noted what's wrong in Tadjbakhsh and Keller's work. Also they presented bimodal formulation of the column problem. Namely, structural optimization combines mathematics and mechanics with engineering and has become a broad multidisciplinary field, see Prager and Taylor [6], Rozvany and Mroz [7] and Vanderplaats [8]. The constant flow of general reviews, surveys of subfields and conference proceedings on optimal design testify the strong activity and increasing importance of the field, see Olhoff and Taylor [9] as a source for additional bibliography. We note that the problem of Lagrange and the optimal shapes for which the cross sections vanishes at certain points were reconsidered many times, see results of Barnes [10], [11], Seiranian [12], Plaut *et al.* [13] and Cox and Overton [14]. Now it is clear that all these results are mathematically correct. Also it is remarkable that these results were based on the linear equilibrium equations that follow from Bernoulli-Euler bending theory as the simplest model for the planar flexure of an elastic column described by an inextensible curve constrained to lie in plane.

Optimization of columns for maximum buckling load still remains a topic of widespread interest. In what follows, the problem of Lagrange is generalized in two directions. First, the optimal shape of compressed column that can suffer not only flexure as in the classical elastica theory, but also compression and shear will be considered. Also, by use of Kirchhoff's kinetic analogue, see Antman and Kenney [15], the optimal shape of compressed and twisted column against spatial buckling will be determined. We shall show that both solutions that follow from these two possible generalizations of the classical Bernoulli-Euler elastica theory lead to the optimum columns with non-zero cross sectional area at the ends. Namely, according to Clausen's solution the optimum column has zero cross sectional area at the ends, and therefore, roughly speaking, at the ends does not recognize the difference between any applied finite load and, for example, its doubled value. It seems that Clausen's solution will never fit for real columns. In considering Nikolai's work [1] and his comments on that solution, three points should be noted. First, the condition that limiting stress should be included in the column problem is taken from the linear theory called strength of materials. Secondly, it was implicitly allowed that the strongest column will change its length under compression. Thus, Nikolai's solution could not be considered within Bernoulli-Euler beam theory because the classical theory of buckling neglects axial strain of the central line as a possible deformation of an elastic rod. Lastly, although according to Gjelsvik [16], the first work that generalizes the classical elastica theory as to take the shearing forces into account goes back to Engesser, who considered the influence of shear on the buckling loads in 1889, in the problem of Lagrange the infinite value of shear rigidity was assumed.

The aim of any theory of rods or beams is to describe the deformed configuration of a slender three-dimensional body by a single curve and certain parameters recording material orientation relative to that curve, see Parker [17]. The three-dimensional elastic constitutive law is replaced by expressions for resultant forces, moments and generalized moments in terms of extension, curvatures, torsion and the remaining parameters. Each resulting theory must necessarily be approximate, although its accuracy should increase as the representative scale of distance along the axis of the rod increases relative to a typical diameter of the cross section. According to that lines we find that Clausen's solution is correct in sense of the classical Bernoulli-Euler elastica theory, and if we would like to improve it to fit for real columns we are to pose the column problem within a generalized plane elastica theory. In doing that both effects, shear and compressibility, should be taken into account. Namely, it is a well known fact that the Euler buckling load is sensitive of both effects. In other words, decreasing of shear rigidity of the column the value of the Euler buckling load decreases, and the decreasing of the extensional rigidity the value of critical load increases. Thus, in the first part we intend to consider effects of shear and compressibility on the optimal shape of the compressed column.

The next problem, we consider here, is motivated by Keller's work [3] and is suggested by adopting a slightly different generalization of the classical Bernoulli-Euler elastica. With a plausible generalization of the column load that leads to spatial buckling we intend to make another study of the Lagrange problem. We shall add twisting couples at the ends of the column for which, once again, the infinite values for shear and extensional rigidity will be assumed. Namely, Grenhhill in 1883 studied the buckling of a column under terminal thrust and torsion, and determined Euler buckling load for the column of uniform circular cross section, see Love[18]. For that load the distribution of material along the length of compressed and twisted column will be optimized so that the column will be of minimum volume and will support a given load without spatial buckling.

In order to motivate the approach to be followed, let us recall formulations presented in papers Chung and Zung [19] and Bratus and Zharov [20]. Also, we shall recall Pearson's formulation of Lagrange problem [14], that is, *to find the curve which by its revolution about an axis in its plane determines the column of great efficiency,* and note that efficiency here denotes structure's resistance to buckling under axial compression. In what follows only a symmetric buckling mode within the classical Bernoulli-Euler elastica theory will be considered. Necessary conditions for optimal control problems we treat here will be derived using the maximum principle of Pontryagin, see Kirk [21] or Alekseev *et al.* [22]. The optimal shapes will be obtained by numerical integration.

Let us consider a slender column represented in rectangular Cartesian coordinate system Ozx by a plane curve C of a given length L as shown in Figure 1. The curve C represents the column axis which coincides with the centroidal line of the column cross sections. In order to deal with a well defined problem we assume that the column is hinged at either end, the hinge at the origin O being fixed whereas the other one is free to move along the axis z. We assume that the column is loaded by a concentrated force F having the action line along the z axis which coincide with the column axis in the undeformed state. Namely, Figure 1 shows the deformation of the column under its buckling load. We denote the bending rigidity of the column and the angle between the tangent to the column axis and the z axis with $EI = EI\,(S)$ and $\psi = \psi\,(S)$ respectively. Here S denotes the arc length of C measured from one end point. Implicitly it is assumed here that extensional and shear rigidity are infinite.

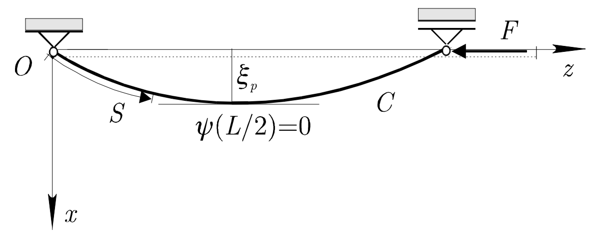

Figure 1. Hinged column under buckling load.

We note that according to Pearson's formulation, the column is of circular cross section, of area $A = A(S)$, so that either $EI(S)$ or $A(S)$ determines the distribution of material along the length of a column. For a uniform column these values are EI_o and A_o respectively.

The half volume of the column is

$$V = \int_0^{L/2} A(S)\, dS. \tag{1}$$

It is well known fact that the equilibrium configuration of the column is given by the following differential equations

$$x' = \sin\psi, \qquad z' = \cos\psi, \quad \psi' = -\frac{Fx}{EI}, \tag{2}$$

where $(\cdot)' = d(\cdot)/dS$, and the following boundary conditions

$$x(0) = 0, \quad z(0) = 0, \quad \psi(L/2) = 0. \tag{3}$$

Also, the critical load for the case of compressed column of uniform cross section bounded as shown in Figure 1 is

$$F_c = \frac{\pi^2 EI_o}{L^2}, \tag{4}$$

for example see Love [18]. Note that z variable can be omitted from the following analysis.

We introduce now the following non-dimensional quantities

$$t = \frac{S}{L}, \quad \xi = \frac{x}{L}, \quad a = \frac{A}{A_o}, \quad v = \frac{V}{A_o L}, \quad \lambda = \frac{FL^2}{EI_o}, \tag{5}$$

and note

$$\frac{EI}{EI_o} = \frac{\frac{EA^2}{4\pi}}{\frac{EA_o^2}{4\pi}} = a^2.$$

Then Eqs. (1), (2) and (3) become

$$v = \int_0^{1/2} a\, dt, \tag{6}$$

$$\dot{\xi} = \sin\psi, \qquad \dot{\psi} = -\frac{\lambda\xi}{a^2}, \tag{7}$$

$$\xi(0) = 0, \quad \psi(1/2) = 0. \tag{8}$$

where $\dot{(\cdot)} = d(\cdot)/dt$. From now on we choose $a = a(t)$ to be the non-dimensional parameter that determines the distribution of material along the column axis.

Now, following the lines of the above papers [19], [20] the column's resistance to buckling under axial compression as a problem of optimal control theory could be expressed, at least in three different ways:

I) to find the distribution of material along the length of a column so that the column is of minimum volume and will support a given load without buckling, in our notation, reads

$$\begin{gathered} \min_a v, \\ \dot{\xi} = \psi, \quad \dot{\psi} = -\frac{\lambda\xi}{a^2}, \\ \xi(0) = 0, \quad \psi(1/2) = 0, \end{gathered} \tag{9}$$

II) to find the distribution of material along the length of a column of a given volume which will give the largest possible buckling load, i.e.

$$\begin{gathered} \max_a \lambda, \\ \dot{\xi} = \psi, \quad \dot{\psi} = -\frac{\lambda\xi}{a^2}, \quad \dot{v} = a, \quad \dot{\lambda} = 0, \\ \xi(0) = 0, \quad \psi(1/2) = 0, \quad v(0) = 0, \quad v(1/2) = v_p, \end{gathered} \tag{10}$$

where v_p is that given volume, and

III) to find the distribution of material along the length of a column of a given volume which will give the largest possible load provided given postbuckling deformation will not be exceeded, i.e.

$$\begin{gathered} \max_a \lambda, \\ \dot{\xi} = \sin\psi, \quad \dot{\psi} = -\frac{\lambda\xi}{a^2}, \quad \dot{v} = a, \quad \dot{\lambda} = 0, \\ \xi(0) = 0, \quad \psi(1/2) = 0, \quad v(0) = 0, \quad v(1/2) = v_p, \quad \xi(1/2) = \xi_p, \end{gathered} \tag{11}$$

where ξ_p is given as the maximum deflection of the column in the post-critical region.

We note that in the third formulation the nonlinear equilibrium equations are used. Also for small values of ξ_p the second and the third formulation coincide. In solving problems given by Eqs. (9) - (11) Pontryagin's maximum principle is used. According to that principle, necessary conditions for optimality, Euler-Lagrange or costate equations, and natural boundary conditions for the problems (9) - (11) respectively, [21], [22], read

I)

$$\begin{gathered} a = \sqrt[3]{-2\lambda\xi p_\psi}, \\ \dot{p}_\xi = \frac{\lambda p_\psi}{a^2}, \quad \dot{p}_\psi = -p_\xi, \\ p_\xi(1/2) = 0, \quad p_\psi(0) = 0, \end{gathered} \tag{12}$$

II)

$$a=\sqrt[3]{\frac{-2\lambda\xi p_{\psi}}{p_{v}}},$$

$$\dot{p}_{\xi}=\frac{\lambda p_{\psi}}{a^{2}},\quad \dot{p}_{\psi}=-p_{\xi},\quad \dot{p}_{v}=0,\quad \dot{p}_{\lambda}=\frac{p_{\psi}\xi}{a^{2}}, \tag{13}$$

$$p_{\xi}\left(1/2\right)=0,\quad p_{\psi}\left(0\right)=0,\quad p_{\lambda}\left(0\right)=0,\quad p_{\lambda}\left(1/2\right)=1,$$

III)

$$a=\sqrt[3]{\frac{-2\lambda\xi p_{\psi}}{p_{v}}},$$

$$\dot{p}_{\xi}=\frac{\lambda p_{\psi}}{a^{2}},\quad \dot{p}_{\psi}=-p_{\xi}\cos\psi,\quad \dot{p}_{v}=0,\quad \dot{p}_{\lambda}=\frac{p_{\psi}\xi}{a^{2}}, \tag{14}$$

$$p_{\psi}\left(0\right)=0,\quad p_{\lambda}\left(0\right)=0,\quad p_{\lambda}\left(1/2\right)=1,$$

where Lagrange multipliers p_{ξ}, p_{ψ}, p_{v} and p_{λ} and the corresponding Hamiltonians are introduced in the usual way. It is obvious that condition $p_{\psi}(0)=0$ always implies that the cross section vanishes at the end of the column, i.e.,

$$a\left(0\right)=0. \tag{15}$$

Although the obtained problems given by Eqs. (9), (10) and (11) with Eqs. (12), (13) and (14), respectively, could be solved in closed form, as in papers [19], [20], to get the optimal shapes against buckling we shall use numerical methods presented in Press *et al.* [23]. The shooting method is of course a standard technique for solving two point boundary value problems. First we shall examine the optimal column and the uniform column of the same volume. Then, results for the optimal shape obtained from the first problem,with results obtained from the second, and the third formulation, will be compared.

If we consider the buckling load of the slender column of unit length and unite volume with uniform distribution of the material along the column axis, i.e., the critical load $\lambda_{c}=\pi^{2}=9.869$, than the obtained minimum half volume of the optimal column is $v_{\min}=0.433$. In Figure 2 the optimal shape, as a solution of the problem given by Eqs. (9) and (12) is presented. Namely, instead of the solution of the first problem given by Eqs. (9) and (12), say $a=a_{opt}\left(t\right)$, in Figure 2 we present optimal curve obtained as $R_{opt}\left(t\right)=\sqrt{a_{opt}\left(t\right)}$, which according to Pearson formulation, gives the column of greatest efficiency. The obtained shape corresponds to Clausen's solution of Lagrange problem. The uniform column of the same volume as the optimal one with corresponding constant radius of the cross section $R_{1}=0.93<1$ is also shown.

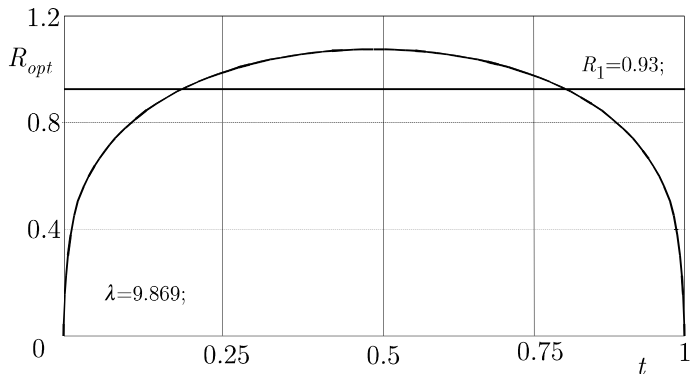

Figure 2. Clausen's solution and uniform column.

In order to examine the postbuckling behavior of the column of unit length and the half volume $v_1 = 0.433$ with uniform cross section first, we note that its buckling load is $\lambda_1 = 7.402 < \lambda_c$. Further, we solve two point boundary value problem given by Eqs. (7), (8) with $\lambda = \lambda_c$ and $a_1^2 = 0.75$. In Figure 3 the post-critical shape of the curve C for the uniform column of half volume $v_1 = 0.433$ is shown.

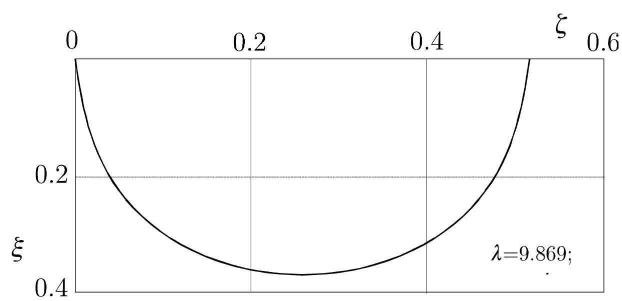

Figure 3. Postcritical shape of the uniform column.

We note very large deflection of the uniform column axis. When compared with the uniform column, the optimal column loaded with $\lambda = \lambda_c$ still remains straight. It may be added that the optimal shapes for the first and the second formulation are the same. Namely, if we choose $v_p = 0.433$ and solve the second problem given by Eqs. (10) and (13) we get $\lambda_{\max} = \pi^2$ as expected. Thus, the first and the second formulation are equivalent, but the first one is easier to tackle. Also, if we put the value that corresponds to the column of unit length and unite volume, $v_p = 0.5$, from the second problem we obtain $\lambda_{\max} = 13.159$. Similarly, with value $\lambda = 13.159$ according to the first formulation the minimum half volume of the optimal column equals $v_{\min} = 0.5$.

To investigate the agreement of the second and the third solution first we need to examine the value ξ_p. In Bratus and Zharov [20], where instead of $\xi_p = \xi\,(1/2)$ the value $\psi\,(0)$ is proposed, very large deflections in post buckling region are allowed. Some of them are even bigger then the half length of the column. A first observation is that in engineering we intend to keep column in trivial - straight position so we propose here the values of ξ_p to be not more than 15% of the column length. If we solve the third problem given by Eqs. (11) and (14) with $v_p = 0.5$ and $\xi_p = 0.102$ we get $\lambda_{\max} = 13.425$. As expected the decreasing of the value ξ_p the closeness of the second and the third solution increases. For example, with $v_p = 0.5$ and $\xi_p = 0.05$ we get $\lambda_{\max} = 13.221$. Also we find that the difference between solution for $a_{opt}(t)$ obtained from the second (linear) problem, and the corresponding solution of the third (nonlinear) problem, is less then 10^{-2}.

In conclusion of this section we claim that among the equivalent problems the first one is optimal for engineering applications. Namely, the previous analysis shows that the first formulation corresponds to the problem of minimum dimension. Also, the first formulation is based the linear equilibrium equations. We note that, the considerations concerning boundary condition ξ_p in the third one will motivate our approach in spatial buckling problem. Our next step is to analyze the column problem within a generalized plane elastica theory with axial and shear deformations.

2 The plane problem

In the analysis that follows we shall examine the problem of Lagrange with the generalized constitutive equations given in Atanackovic and Spasic [24]. In other words, we intend to consider the same load configuration and the symmetric buckling mode as shown in Figure 1, but now with finite values for extensional and shear rigidity. As before we assume that the column is straight in unload state. First, we shall derive nonlinear equilibrium equations. Then we shall show that the bifurcation points of the nonlinear equilibrium equations are determined by the eigenvalues of the linearized equations. Finally, we shall determine the optimal shape for the column model based on simple shear of finite amount [24].

2.1 Differential equations and boundary conditions

An element of the column axis whose length in the undeformed state is dS in the deformed state has the length ds.

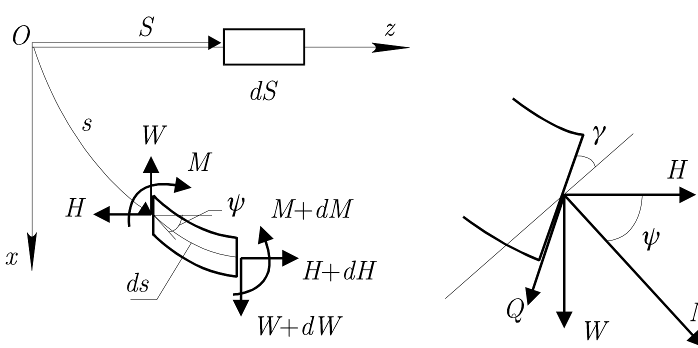

Figure 4. Geometry of the deformed column element and components of the contact force.

From Figure 4 we get the equilibrium equations for an element of length ds in the deformed state

$$dH = 0, \quad dW = 0, \tag{16}$$

$$dM = W dz - H dx, \tag{17}$$

where H and W are z and x components of the resultant force respectively, M is the resultant couple, z and x are the coordinates of an arbitrary point on the column in the deformed state. The strain of the central axis is defined as

$$\varepsilon = \frac{ds - dS}{dS}. \tag{18}$$

Referring again to Figure 4 the following geometrical relations hold

$$dz = (1+\varepsilon)\cos\psi dS, \quad dx = (1+\varepsilon)\sin\psi dS, \tag{19}$$

where, as above, ψ is the angle between the tangent to the column axis and the z axis, and where we used Eq. (18). Note that from Figure 1 the following physical and geometrical boundary conditions corresponding to Eqs. (16), (17) and (19) read

$$H(0) = -F, \quad W(0) = 0, \tag{20}$$

$$M(0) = 0. \tag{21}$$

$$z(0) = 0, \quad x(0) = 0, \tag{22}$$

$$\psi(L/2) = 0. \tag{23}$$

Finally, we take constitutive equations of the column in the form that follow from Atanackovic and Spasic [24]

$$M = -EI\left(\frac{d\psi}{dS} - \frac{d\gamma}{dS}\right), \tag{24}$$

$$\sin\gamma = k\frac{Q}{GA}, \tag{25}$$

$$\varepsilon = \frac{N}{EA}, \tag{26}$$

where γ is the shear angle, Q is the component of the contact force in the direction of sheared planes (that is, Q has angle $\pi/2+\gamma$ with the tangent to the column axis), GA is the shear rigidity, k is the shear correction factor that depends on the geometry of the cross section and on the material, see Renton [25], N is the component of the contact force in the direction of the tangent to the column axis and EA is the extensional rigidity. In this equations we recognize Engesser's model of decomposition in which the internal forces in an arbitrary cross section of the column are decomposed into non orthogonal directions, see Gjelsvik [16]. From Figure 4 we conclude that

$$N = H\frac{\cos(\psi-\gamma)}{\cos\gamma} + W\frac{\sin(\psi-\gamma)}{\cos\gamma}, \quad Q = W\frac{\cos\psi}{\cos\gamma} - H\frac{\sin\psi}{\cos\gamma}. \tag{27}$$

Integrating Eqs. (16) and using boundary conditions given by Eqs. (20) we get

$$H = -F, \quad W = 0. \tag{28}$$

Now Eqs. (25) and (26) become

$$\sin 2\gamma = \frac{2kF}{GA}\sin\psi, \tag{29}$$

$$\varepsilon = -\frac{F}{EA}\frac{\cos(\psi-\gamma)}{\cos\gamma}, \tag{30}$$

where we used Eq. (27). From Eq. (29) we can express γ in terms of ψ

$$\gamma = \frac{1}{2}\arcsin\left(\frac{2kF}{GA}\sin\psi\right). \tag{31}$$

Now we define a new variable

$$\alpha = \psi - \gamma, \tag{32}$$

or

$$\alpha = \psi - \frac{1}{2}\arcsin\left(\frac{2kF}{GA}\sin\psi\right), \tag{33}$$

and note that the application of the implicit function theorem to Eq. (33) ensures the existence of the following relation

$$\psi = f(\alpha), \tag{34}$$

at least near zero. Then Eqs. (24) and (17) become

$$M' = F\left(1 - \frac{F}{EA}\frac{\cos\alpha}{\cos\left\{\frac{1}{2}\arcsin\left[\frac{2kF}{GA}\sin f(\alpha)\right]\right\}}\right)\sin f(\alpha), \quad \alpha' = -\frac{M}{EI}, \tag{35}$$

where $(\cdot)' = d(\cdot)/dS$. The boundary conditions corresponding to Eqs. (35) are

$$M(0) = 0, \quad \alpha(1/2) = 0, \tag{36}$$

where we used Eqs. (23), (29) and (32). Now the equilibrium configuration of the column is described.

The linearized boundary value problem corresponding to Eqs. (35), (36) leads to equations

$$M' = F\frac{1 - \dfrac{F}{EA}}{1 - \dfrac{kF}{GA}}\alpha, \quad \alpha' = -\frac{M}{EI}, \tag{37}$$

with boundary conditions (36). Namely, if we linearize Eq. (29) we get

$$\gamma = \frac{kF}{GA}\psi,$$

and now Eq. (34) becomes

$$\psi = \frac{1}{1 - \dfrac{kF}{GA}}\alpha.$$

We note that for $EA,\ GA \to \infty$, we could easily get the linearization of equilibrium equations for the classical column model given by Eqs. (2) and (3).

2.2 Bifurcation of the trivial solution

To determine the critical load for the uniform column with constant cross section of area A_o, we shall use the methods presented in Krasnosel'skii *et al.* [26]. Namely, to prove that eigenvalues of the linearized problem determine bifurcation points of the nonlinear equilibrium equations we shall use the fact that the eigenvalues of the linearized equilibrium equations are stable in a specially defined sense.

First, we note that variables z and x can be omitted from bifurcation analysis since they could be determined after we find $\alpha(S)$ and then $\psi(S)$ and $\gamma(S)$. Then, we add more the non-dimensional quantities to Eqs. (5). Namely, we introduce

$$\beta = \frac{kF}{GA_o}, \quad \mu = \frac{F}{EA_o}, \quad m = \frac{ML}{EI_o}, \tag{38}$$

and note that the nonlinear equilibrium equations of the uniform column with corresponding boundary conditions could be written in the following form

$$\dot{m} = \lambda \left\{ 1 - \mu \frac{\cos\alpha}{\cos\left\{\frac{1}{2}\arcsin\left[2\beta\sin f(\alpha)\right]\right\}} \right\} \sin f(\alpha), \quad \dot{\alpha} = -m, \tag{39}$$

$$m(0) = 0, \quad \alpha(1/2) = 0, \tag{40}$$

where $\dot{(\cdot)} = d(\cdot)/dt$ and where we used Eqs. (38) and (5). For all values of λ system (39) with boundary conditions (40) admits a trivial solution

$$m \equiv 0, \quad \alpha \equiv 0,$$

in which the column axis remains straight, see Eq. (32) as well as Eq. (29). Namely, we treat λ as the bifurcation parameter; that is, we fix F, L, EA_o, and GA_o and allow bending rigidity EI_o to vary. Our next goal is to find the smallest value of λ, denoted by λ_{cg} for which boundary value problem (39), (40) has non-trivial solution. Concerning that system we note that the right hand sizes of Eqs. (39) do have continuous derivatives with respect to m and α for $t \in [0,1]$, m and α bounded, say $m^2 + \alpha^2 < \sigma^2 \in R^+$, for $\lambda > 0$, and do vanishes on the trivial solution.

The linearized boundary value problem corresponding to Eqs. (39), (40) reads

$$\dot{m} = \lambda \frac{1-\mu}{1-\beta}\alpha, \quad \dot{\alpha} = -m, \tag{41}$$

with boundary conditions (40). In system (41) we used Eqs. (37), (38) and (5). We transform Eqs. (41) to

$$\ddot{m} + \kappa^2 m = 0,$$

where $\kappa^2 = \lambda(1-\mu)/(1-\beta)$, and then find the solution of the linearized problem (41) in the following form

$$m = D_n \sin\kappa_n t, \quad \alpha = \frac{D\cos\kappa_n t}{k_n}, \quad n = 1, 2, ... \tag{42}$$

where D_n are constants and where we used the first boundary condition (38). The condition that determines the critical load in sense of generalized elastica, λ_{cg} reads

$$\cos\frac{\kappa}{2} = 0 = \cos\frac{\pi}{2},$$

so we get

$$\lambda_{cg} = \pi^2 \frac{(1-\beta)}{(1-\mu)}. \tag{43}$$

We note that for $\beta = \mu = 0$, we recover the classical critical load $\lambda_{cg} = \lambda_c$ as expected. Also, for $\mu = 0$ we recover the result of Gjelsvik [26]. From Eq. (43) we see the well known fact that shear and compressibility have the opposite influence on Euler buckling load. According to Renton [25], here and in all numerical examples that follows, for real columns we shall assume that $\mu < 1$ and that β is greater then μ.

Finally, to prove that the eigenvalues of linearized equations (41), (40) determine the bifurcation points of nonlinear equations (39), (40) it is necessary to show that the eigenvalues of (41), (40) are stable in a specially defined sense, see Krasnosel'skii *et al.* [26]. To recognize the stable eigenvalue, say λ_s, first we define a function $\chi(t,\lambda)$ by relation

$$\tan\chi(t,\lambda) = \frac{\alpha(t,\lambda)}{m(t,\lambda)},$$

where $m(t,\lambda)$ and $\alpha(t,\lambda)$ are given by Eqs. (42). Then we require the condition

$$\frac{d\left[\chi(1/2,\lambda)\right]}{d\lambda}\Big|_{\lambda=\lambda_s} \neq 0,$$

to be satisfied. Namely, the condition that the function $\chi(1/2,\lambda)$ does not attain its local extremum at λ_s ensures the stability of an eigenvalue in the sense of Krasnosel'skii *et al.* [26]. For $n = 1$, λ_{cg} given by Eq. (43) and for m and ψ given by Eqs. (42) that derivative reads

$$\frac{d\left[\chi(1/2,\lambda)\right]}{d\lambda}\Big|_{\lambda=\lambda cg} = -\frac{1}{4}\frac{-1+\mu}{\pi^2(-1+\beta)} \neq 0.$$

This analysis confirms that bifurcation takes place at the characteristic values of linearized equations.

We make two remarks here. First, the proof that the eigenvalues of the linearized equilibrium equations (41), (40) are bifurcation points of the full nonlinear equilibrium equations (39), (40) could also proceed as either on applying the standard procedure of Liapunov-Schmidt reduction, see Chow and Hale [27] and Troger and Steindl [28] or on rewriting the governing differential equations as an integral equation and on applying degree theoretic arguments of local bifurcation theory, see Antman and Rosenfeld [29] and Hutson and Pym [30]. Both procedures assert that if the multiplicity of a characteristic value of linearized problem is odd, bifurcation does indeed take place. Second, note that instead of Engesser's model we could use either Haringx's or Timoshenko type of decomposition, see Atanackovic [31]. Namely, the linearized form of Eqs. (24) - (26) will be the same but Eqs. (27) will differ because the resultant force could be decomposed in either the direction of the shared cross section and into the direction normal to the sheared cross section, as in Reissner [32], and Goto *et al.* [33], or in the direction of the column axis and the direction orthogonal to the column axis, as in DaDeppo and Schmidt [34]. Namely, it is well known fact that the critical load depends on the type of decomposition, see Gjelsvik [16] and Atanackovic and Spasic [24], and it seems reasonably to expect that the optimal shape of the column in sense of a generalized elastica will also depend on the type of decomposition i.e., on the adopted column model.

In order to examine the influence of the finite values of extensional and shear rigidity, on the optimal shape of compressed column we shall rewrite the linearized equations (37) in the following form

$$M' = F\frac{1-\dfrac{\dfrac{F}{EA_o}}{\dfrac{EA}{EA_o}}}{1-\dfrac{\dfrac{kF}{GA_o}}{\dfrac{GA}{GA_o}}}\alpha, \quad \alpha' = -\frac{\dfrac{M}{EI_o}}{\dfrac{EI}{EI_o}},$$

or in the dimensionless form as

$$\dot{m} = \lambda\left(\frac{a-\mu}{a-\beta}\right)\alpha, \quad \dot{\alpha} = -\frac{m}{a^2},$$

where we used Eqs. (5) and (38). The corresponding boundary conditions are given by Eqs. (40).

2.3 Optimal control problem

With the preparations of the previous section complete, we are now in a position to generalize the column problem denoted by I, that is to generalize formulations given by Eqs. (9) and (12). Namely, we pose the following problem

IV) to find the distribution of material along the length of a column, for given β, $\mu > 0$, so that the column is of minimum volume and will support a given load $\lambda = \lambda_{cg}$, given by Eq. (43), without buckling, i.e.,

$$\min_a v,$$

$$\dot{m} = \lambda\left(\frac{a-\mu}{a-\beta}\right)\alpha, \quad \dot{\alpha} = -\frac{m}{a^2}, \tag{44}$$

$$m(0) = 0, \quad \alpha(1/2) = 0,$$

where v is given by Eq. (6).

According to Pontryagin maximum principle we introduce Lagrange multipliers p_m and p_α to form Hamiltonian

$$H = a + p_m\lambda\left(\frac{a-\mu}{a-\beta}\right)\alpha - p_\alpha\frac{m}{a^2}.$$

Then, the optimal distribution of the material a is determined from the relation

$$\frac{\partial H}{\partial a} = 0 = 1 + \frac{2p_\alpha m}{a^3} + \frac{\lambda \alpha p_m (\mu - \beta)}{(a - \beta)^2}. \tag{45}$$

The corresponding costate equations and natural boundary conditions are

$$\dot{p}_m = -\frac{\partial H}{\partial m} = \frac{p_\alpha}{a^2}, \quad \dot{p}_\alpha = -\frac{\partial H}{\partial \alpha} = -p_m \lambda \left(\frac{a - \mu}{a - \beta} \right), \tag{46}$$

$$p_m (1/2) = 0, \quad p_\alpha (0) = 0. \tag{47}$$

We note that from Eqs. (45) and (47) it follows

$$a (0) = \beta + \sqrt{\lambda (\beta - \mu) \alpha (0) p_m (0)}, \tag{48}$$

as a new result that generalizes the one given by Eq. (15). According to Eq. (48) the shape of the optimal column at its end depends on the load and material. In other words the optimal shape becomes sensitive of the load and it seems it will fit for real columns.

In order to get the optimal shape for several values of β and μ we calculate Euler buckling load first $\lambda = \lambda_{cg}$ and then we solve two point boundary value problem (44), (46) and (47) with a obtained as a solution of Eq. (45). At each step of integrating procedure Eq. (45) is solved numerically by bisection method. All numerical integrations for shooting method were based on Bulirsch-Stoer integration procedure [23]. In Table 1. we present minimum half volume $v_{\min}$, the minimal and the maximal area of the column cross section, that is, $a_{\min} = a (0)$ and $a_{\max} = a (1/2)$. For comparison, in Table 1 Clausen solution, that corresponds to zero values of β and μ, and the critical value $\lambda_c = \pi^2 = 9.869$, here obtained by numerical solution of the column problem. as a special case is also presented.

Table 1.

β	μ	λ	$v_{\min}$	$a_{\max} = a (1/2)$	$a_{\min} = a (0)$
0.0	0.0	9.869	0.433	1.1547	0
0.0003	0.0001	9.868	0.433	1.1548	0.0187
0.003	0.001	9.850	0.434	1.1552	0.0598
0.03	0.01	9.670	0.440	1.1584	0.1959
0.3	0.1	7.676	0.476	1.1471	0.6693

The optimal shapes of the column that correspond to several values of column parameters obtained by numerical integration of two point boundary value problem (44), (46) and (47), with a obtained as a solution of Eq. (45), are presented in Figure 5. As before, in Figure 5, we present optimal curve obtained as $R_{opt} (t) = \sqrt{a_{opt} (t)}$, which according to Pearson formulation of the Lagrange problem, gives the column of greatest efficiency.

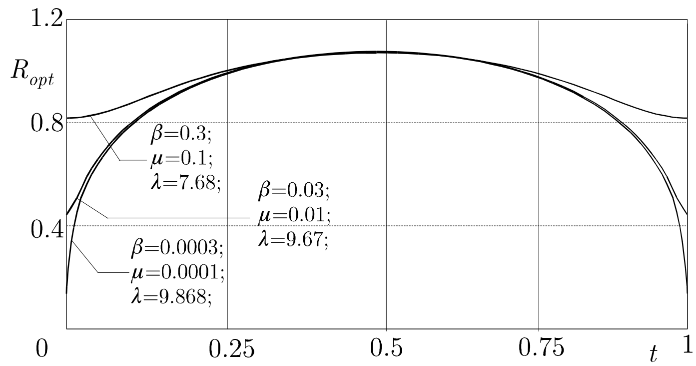

Figure 5. Optimal shapes that generalize Clausen's solution.

We note that finite values of shear and extensional rigidity did change Clausen's solution as expected. We conclude by noting that, along the lines of Keller [3] or Atanackovic [35], the results presented here could be generalized by introducing imperfections in the shape and loading and by allowing different boundary conditions. An interesting type of boundary conditions, applicable to optimal design of water tower is presented in Spasic and Glavardanov [36].

3 Optimal shape of the column against spatial buckling

In this section we wish to consider the buckling of a column subjected to a twisting couple and compressive axial load. The ends of the column are assumed to be attached to the supports by ideal spherical hinges and are free to rotate in any directions. As before one hinge is fixed whereas the other one is free to move along the axis z. We assume that the column is loaded by a concentrated force F and a twisting couple K having the action line along the z axis which coincide with the column axis in the unloaded state in which the column is straight and prismatic, see Figure 6.

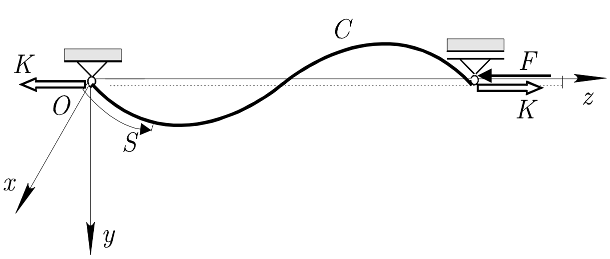

Figure 6. Load configuration and spatially buckled column.

Also, let us assume that during buckling the force and the couples K retain their initial directions. Although under such conditions the couple K is non conservative, see Timoshenko and Gere [37], we intend to apply Euler method of stability analysis or the method of adjacent equilibrium. In this case the deflection curve C will not be a plane curve.

3.1 Equilibrium equations of a compressed and twisted column

As before, the column under consideration is of circular cross section and will be represented by the incompressible curve of centers C and circles attached to each point of that curve. In what follows we introduce the line joining centroids of circular cross sections as a space curve C. We can represent that curve with respect to a reference frame $Oxyz$ with $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ as the corresponding unit vectors, fixed at the origin O, by means of a vector

$$\mathbf{r} = x(S)\,\mathbf{i} + y(S)\,\mathbf{j} + z(S)\,\mathbf{k},$$

where S is the Lagrange coordinate as usual. It is assumed that a unit vector orthogonal to the plane of the circle and the tangent to the column axis at each point of the curve C, coincide. These facts are interpreted as the absence of axial and shear deformations. At the centroid of any circle an orthogonal coordinate system $Cx_1y_1z_1$ is constructed with corresponding unit vectors $\mathbf{i}_1, \mathbf{j}_1$ and $\mathbf{k}_1$. The axis z_1 is orthogonal to circle. In unloaded state unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ and $\mathbf{i}_1, \mathbf{j}_1, \mathbf{k}_1$ coincide, see Figure 7. Then to each circle we attache area $A(S)$ and Euler angles $\psi = \psi(S)$, $\theta = \theta(S)$ and $\varphi = \varphi(S)$, recording orientation of the system $Cx_1y_1z_1$ to the reference frame $Oxyz$. Namely, we introduce a circle of area A, and three spherical angles of Euler type as the parameters recording material orientation to that curve.

Euler angles describe any possible orientation in terms of a rotation ψ abut the y axis, then a rotation θ about the new x axis, and finally, a rotation about new z axis of φ. According to that sequence of rotations the following geometrical relations between unit vectors $\mathbf{i}_1, \mathbf{j}_1$ and $\mathbf{k}_1$ and unit vectors $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ hold, see Atanackovic [31],

$$\begin{aligned} \mathbf{i}_1 &= (\cos\psi\cos\varphi + \sin\psi\sin\theta\sin\varphi)\,\mathbf{i} + \cos\vartheta\sin\varphi\;\mathbf{j} + (\cos\psi\sin\theta\sin\varphi - \sin\psi\cos\varphi)\,\mathbf{k}, \\ \mathbf{j}_1 &= (\sin\psi\sin\theta\cos\varphi - \cos\psi\sin\varphi)\,\mathbf{i} + \cos\theta\cos\varphi\;\mathbf{j} + (\sin\psi\sin\varphi + \cos\psi\sin\theta\cos\varphi)\,\mathbf{k}, \\ \mathbf{k}_1 &= \sin\psi\cos\theta\mathbf{i} - \sin\theta\;\mathbf{j} + \cos\psi\cos\theta\mathbf{k}. \end{aligned} \tag{49}$$

Also the change of unit vectors $\mathbf{i}_1, \mathbf{j}_1$ and $\mathbf{k}_1$ along the curve C is given by

$$\begin{aligned} \mathbf{i}_1' &= \omega_3\;\mathbf{j}_1 - \omega_2\mathbf{k}_1, \\ \mathbf{j}_1' &= \omega_1\;\mathbf{k}_1 - \omega_3\mathbf{i}_1, \\ \mathbf{k}_1' &= \omega_2\mathbf{i}_1 - \omega_1\mathbf{j}_1, \end{aligned} \tag{50}$$

where $(\cdot)' = d(\cdot)/dS$, and where

$$\begin{aligned} \omega_1 &= \psi'\cos\theta\sin\varphi + \theta'\cos\varphi, \\ \omega_2 &= \psi'\cos\theta\cos\varphi - \theta'\sin\varphi, \\ \omega_3 &= -\psi'\sin\theta + \varphi'. \end{aligned} \tag{51}$$

The elastic deformation of the column can be described by the components of curvature ω_1, ω_2, and ω_3 that represents the twist around the axis z_1. We note here that this type of Euler angles, usually called Krilov or "ship" type, see Lurie[38], is important because the small difference in orientation of the systems $Oxyz$ and $Cx_1y_1z_1$ restricts the value of each of these angles. Also, the fact that this type allows the first two angles to be small and the third one to be finite will be useful latter.

A typical element of the column with force and moment resultants acting on it is shown in Figure 7.

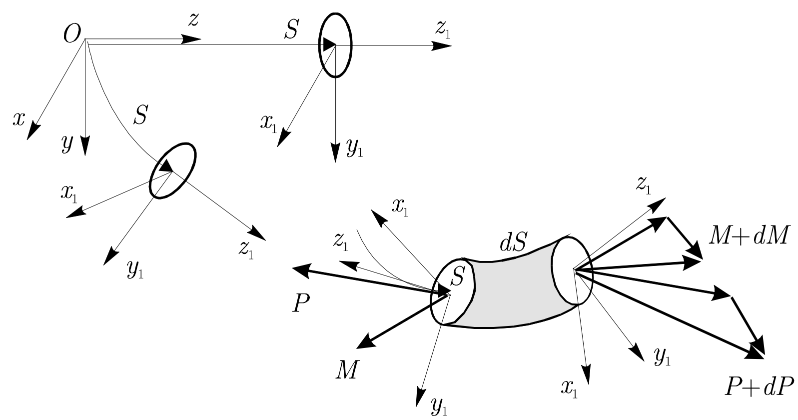

Figure 7. Space curve coordinates and free body diagram of column element.

We assume that there are no static forces or moments distributed along the column element. The condition of force equilibrium applied to the column element yields the following equation

$$\frac{d\mathbf{P}}{dS} = 0. \tag{52}$$

Similar, from consideration of the free body diagram of Figure 7, i.e., summing moments about a point S yields

$$\frac{d\mathbf{M}}{dS} + \frac{d\mathbf{r}}{dS}\times\mathbf{P} = 0, \tag{53}$$

where

$$\frac{d\mathbf{r}}{dS} = x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}. \tag{54}$$

According to the column model it is obvious that $d\mathbf{r}/dS$ and $\mathbf{k}_1$ coincide so the following geometrical relations hold

$$x' = \sin\psi\cos\theta, \quad y' = -\sin\theta, \quad z' = \cos\psi\cos\theta. \tag{55}$$

Finally, assuming linear elastic behavior the connection between the geometrical quantities and the components of the resultant moment

$$\mathbf{M}(S) = M_1\mathbf{i}_1 + M_2\mathbf{j}_1 + M_3\mathbf{k}_1 \tag{56}$$

could be given by

$$M_1 = EI\omega_1, \quad M_2 = EI\omega_2, \quad M_3 = GJ\omega_3, \tag{57}$$

where EI and GJ are bending and torsional rigidity respectively. These constitutive equations form the ordinary approximate theory, a generalization of the classical Bernoulli-Euler plane elastica, and represent the classical Kirchhoff's model of spatially buckled column with no shear and axial deformations, see Eliseyev [39], and no torsion-warping deformation, see Simo and Vu-Quoc [40]. For a column of circular cross section the connection between EI and GJ reads

$$GJ = \frac{EI}{(1+\rho)}, \tag{58}$$

where ρ is the Poisson's ratio. Also, as before EI could be expressed as $EA^2/(4\pi)$.

We state now the boundary conditions corresponding to the column shown in Figure 6. According to the usual sign convention we write

$$\mathbf{P}(L) = -F\mathbf{k}, \tag{59}$$

$$\mathbf{M}(L) = K\mathbf{k}, \tag{60}$$

and note that $\mathbf{P}(0)$ and $\mathbf{M}(0)$ equals $\mathbf{P}(L)$ and $\mathbf{M}(L)$ respectively. We also add the following geometrical conditions

$$x(0) = 0, \quad y(0) = 0, \quad z(0) = 0, \tag{61}$$

$$\varphi(0) = 0, \tag{62}$$

$$x(L) = 0, \quad y(L) = 0. \tag{63}$$

The equations (52), (53) could be interpreted in two different frames either $Oxyz$ or $Cx_1y_1z_1$. In the first case we integrate Eq. (52) with boundary condition (59) and we get

$$\mathbf{P} = \mathbf{P}(S) = -F\mathbf{k}. \tag{64}$$

Also, integrating Eq. (53) with boundary condition (60) gives

$$\mathbf{M} = Fy\mathbf{i} - Fx\mathbf{j} + K\mathbf{k}, \tag{65}$$

where we used Eqs. (64), (54) and (60). In the second case by use of Eqs. (50) we write Eqs. (52) and (53) in the following form

$$\begin{aligned} P_1' - P_2\omega_3 + P_3\omega_2 &= 0, \\ P_2' - P_3\omega_1 + P_1\omega_3 &= 0, \\ P_3' - P_1\omega_2 + P_2\omega_1 &= 0, \end{aligned} \tag{66}$$

$$\begin{aligned} M_1' - M_2\omega_3 + M_3\omega_2 - P_2 &= 0, \\ M_2' - M_3\omega_1 + M_1\omega_3 + P_1 &= 0, \\ M_3' - M_1\omega_2 + M_2\omega_1 &= 0, \end{aligned} \tag{67}$$

that corresponds to a physical space. Substituting Eqs. (57) into Eqs. (67) lead to the well known theory known as Kirchhoff's kinetic analogue, see Nikolai [1] or Love [18]. Namely, the equilibrium equations of a spatially buckled column and Lagrange's coordinate $S \in [0, L]$ are interpreted as the equations of motion of a heavy rigid body turning about fixed point and time respectively. Note, that in optimization theory Eqs. (66) and (67) were first used by Keller [3] who considered the problem of optimal shape against buckling of a naturally straight but twisted column subjected to a compressive load. In his work Keller assumed that no twisting couple is applied. Referring again to Love [18] we note that Euler buckling load for a compressed and twisted column of uniform cross section, determined by Greenhill in 1883, is given by

$$\left(\frac{K}{2EI_o}\right)^2 + \frac{F}{EI_o} = \frac{\pi^2}{L^2}. \tag{68}$$

so the bifurcation analysis of the problem in this section will be omitted.

Finally, the equilibrium equations of the compressed and twisted column could be written in a visual space as well. Namely, we connect Eqs. (56) and (65) by use of Eqs. (49) and then substitute Eqs. (51) and (57). As a result, after some calculations, we get the equilibrium equations of spatially buckled column in the following nonlinear form

$$
\begin{gathered}
x' = \sin\psi\cos\theta, \quad y' = -\sin\theta, \\
z' = \cos\psi\cos\theta, \\
\psi' = \left(\frac{Fy}{EI}\sin\psi + \frac{K}{EI}\cos\psi\right)\tan\theta - \frac{Fx}{EI}, \\
\theta' = \frac{Fy}{EI}\cos\psi - \frac{K}{EI}\sin\psi, \\
\varphi' = \left(\frac{Fy}{GJ}\sin\psi + \frac{K}{GJ}\cos\psi\right)\cos\theta + \left[\frac{Fx}{GJ} + \left(\frac{Fy}{EI}\sin\psi + \frac{K}{EI}\cos\psi\right)\tan\theta - \frac{Fx}{EI}\right]\sin\theta.
\end{gathered} \tag{69}
$$

The corresponding boundary conditions are given by Eqs. (61), (62) and (63).

We note that if, as before, we treat only symmetric buckling mode, in which the resultant moment achieve the maximum value at the middle of the column, that is for $S = L/2$, then instead of Eq. (63) we could use the following relations

$$
y\,(L/2) = 0, \quad \psi\,(L/2) = 0. \tag{70}
$$

Namely, using Eq. (65) and the necessary condition

$$
\frac{d\,|\mathbf{M}|}{dS} = \frac{F\,(xx' + yy')}{\sqrt{F^2\,(x^2 + y^2) + K^2}} = 0,
$$

together with Eqs. (55) we get Eqs. (70). Note that for the case $K = 0$, which implies $y = 0$ and $\theta = 0$, we recover Eqs. (2).

As we did in the introduction, we shall use the full nonlinear equations, given by Eqs. (69), to examine the postbuckling behavior of the column of uniform cross section and the same volume as the optimal one. For that purpose, we shall cast the problem into non dimensional form. Namely, we define

$$
\nu = \frac{KL}{EI_o}, \quad \eta = \frac{y}{L}, \quad \zeta = \frac{z}{L}, \tag{71}
$$

and note

$$
\frac{GJ_o}{EI_o} = \frac{1}{(1+\rho)}.
$$

Then we rewrite Eqs. (69) in the following form

$$
\begin{gathered}
\dot{\xi} = \sin\psi\cos\theta, \quad \dot{\eta} = -\sin\theta, \\
\dot{\zeta} = \cos\psi\cos\theta, \\
\dot{\psi} = \frac{1}{a^2}\left[(\lambda\eta\sin\psi + \nu\cos\psi)\tan\theta - \lambda\xi\right],
\end{gathered} \tag{72}
$$

$$\dot{\theta} = \frac{1}{a^2}\left(\lambda\eta\cos\psi - \nu\sin\psi\right),$$

$$\dot{\varphi} = \frac{1}{a^2}\left\{(1+\rho)\left(\nu\cos\psi + \lambda\eta\sin\psi\right)\cos\theta + \left[\lambda\rho\xi + \left(\lambda\eta\sin\psi + \nu\cos\psi\right)\tan\theta\right]\sin\theta\right\},$$

where we used Eqs. (5) as well. The corresponding boundary conditions read

$$\xi(0) = 0, \quad \eta(0) = 0, \quad \zeta(0) = 0, \quad \varphi(0) = 0, \quad \eta(1/2) = 0, \quad \psi(1/2) = 0. \tag{73}$$

The two point boundary problem (72), (73) admits a trivial solution in which the column remains straight but twisted

$$\xi \equiv 0, \quad \eta \equiv 0, \quad \zeta = t, \quad \psi \equiv 0, \quad \theta \equiv 0, \quad \varphi = \frac{(1+\rho)\,\nu}{a^2}t,$$

where the value of twisting angle φ is not necessary small, but is finite.

3.2 Formulation of the problem

In order to determine the optimal shape, we shall linearize the nonlinear equilibrium equations. For the type of Euler angles we use here, the linearized equations are simply obtained from Eqs. (72) as

$$\begin{gathered}\dot{\xi} = \psi, \quad \dot{\eta} = -\theta, \\ \dot{\psi} = \frac{\nu\theta - \lambda\xi}{a^2}, \quad \dot{\theta} = \frac{\lambda\eta - \nu\psi}{a^2}, \\ \dot{\varphi} = \frac{\nu(1+\rho)}{a^2},\end{gathered} \tag{74}$$

where ζ variable is omitted. The corresponding boundary conditions are

$$\xi(0) = 0, \quad \eta(0) = 0, \quad \varphi(0) = 0, \quad \eta(1/2) = 0, \quad \psi(1/2) = 0. \tag{75}$$

Also, from Eq. (68) that determines the critical load for a uniform column, transformed into non dimensional form, we can express dimensionless force $\lambda = FL^2/EI_o$ in terms of dimensionless twisting couple $\nu = KL/EI_o$ as

$$\lambda = \pi^2 - \frac{\nu^2}{4}. \tag{76}$$

Now we are ready to pose another problem that generalizes formulation given by Eqs. (9) and (12). Namely, we consider the problem

V) to find the distribution of material along the length of a column, so that the column is of minimum volume and will support a load for given ν and λ determined from (76), without buckling, i.e.,

$$\min_a v,$$

$$\dot{\xi} = \psi, \quad \dot{\eta} = -\theta, \quad \dot{\psi} = \frac{\nu\theta - \lambda\xi}{a^2}, \quad \dot{\theta} = \frac{\lambda\eta - \nu\psi}{a^2}, \quad \dot{\varphi} = \frac{(1+\rho)\,\nu}{a^2}, \tag{77}$$

$$\xi(0) = 0, \quad \eta(0) = 0, \quad \varphi(0) = 0, \quad \eta(1/2) = 0, \quad \psi(1/2) = 0.$$

where $\dot{(\cdot)} = d(\cdot)/dt$ and where v is given by Eq. (6).

Now we introduce Lagrange multipliers $p_\xi,\ p_\eta,\ p_\psi,\ p_\theta$ and p_φ to form Hamiltonian

$$H = a + p_\xi \psi - p_\eta \theta + p_\psi \frac{\nu\theta - \lambda\xi}{a^2} + p_\theta \frac{\lambda\eta - \nu\psi}{a^2} + p_\varphi \frac{(1+\rho)\,\nu}{a^2}.$$

Then, according to Pontryagin's maximum principle we find the optimal distribution of the material a as a solution of the following equation

$$\frac{\partial H}{\partial a} = 0 = 1 - \frac{2\left[p_\psi(\nu\theta - \lambda\xi) + p_\theta(\lambda\eta - \nu\psi) + p_\varphi \nu (1+\rho)\right]}{a^3}, \tag{78}$$

that is

$$a_{opt} = \sqrt[3]{2\left[p_\psi(\nu\theta - \lambda\xi) + p_\theta(\lambda\eta - \nu\psi) + p_\varphi \nu (1+\rho)\right]}. \tag{79}$$

The corresponding costate equations for this generalization of the column problem are

$$\dot{p}_\xi = \frac{\lambda p_\psi}{a^2}, \quad \dot{p}_\eta = -\frac{\lambda p_\theta}{a^2}, \quad \dot{p}_\psi = -p_\xi + \frac{\nu p_\theta}{a^2}, \quad \dot{p}_\theta = p_\eta - \frac{\nu p_\psi}{a^2}, \quad \dot{p}_\varphi = 0. \tag{80}$$

Finally, the natural boundary conditions read

$$p_\xi(1/2) = 0, \quad ,\ p_\psi(0) = 0, \quad p_\theta(0) = 0, \quad p_\theta(1/2) = 0, \tag{81}$$

and

$$p_\varphi(1/2) = 0. \tag{82}$$

We note that from (81) and (82) follow that $a(0) = 0$. Also that $a(t)$ will be very small in the neighborhood of $t = 0$. For small values of a any activity connected with numerical procedure of solving two point boundary value problem given by Eqs. (77), (79), (80), (81), and (82) is rather complicated because of the stiff equation problem that appears. The situation will be even more complicated if we had chosen to generalize problem III, given by Eqs. (11) and (14), with the nonlinear differential equations as a model. We note that some stiff equations can be handled by a change of variables, see Acton [41], but we shall avoid that problem here on the basis of physical considerations. Namely, if we propose the value of the twist angle at the middle of the column, say

$$\varphi(1/2) = \varphi_p, \tag{83}$$

that is, if we use (83) instead of (82) then we expect the value $p_\varphi = const.$, to differ from zero. Thus, the value $a_{opt}(0)$ given by

$$a_{opt}(0) = \sqrt[3]{2 p_\varphi (1+\rho)\,\nu}, \tag{84}$$

will also differ from zero.

The next problem to be solved is the value of φ_p . Namely, for the uniform column of volume $v = v_1$ in trivial equilibrium configuration, in which the column is straight but twisted, the twist angle at the middle reads

$$\varphi_{1/2} = \frac{(1+\rho)\,\nu}{2}. \tag{85}$$

In the formulation of the optimization problem we expect to get the volume of optimal column, say $v_{\min}$ to be less then v_1 so the value of the twisting angle that corresponds the uniform column of the same volume as optimal, say φ_s will be greater then $\varphi_{1/2}$. Thus, we propose φ_p to be greater then $\varphi_{1/2}$. As a result we expect the values φ_p and φ_s to be close, and that the value of $v_{\min}$ to be in correlation with the difference $\varphi_p - \varphi_s$. Also, for the very small values of the twisting couple ν we expect the optimal shape of the compressed and twisted column to be very close to the Clausen's solution presented in Figure 3.

3.3 Numerical results

In this section we present the results of the numerical integration of the linearized equilibrium equations and necessary conditions for optimality given by Eqs. (77), (79), (80), (81), and (83). As before the shooting method is used. Initial guess $\psi(0)$, $\theta(0)$, $p_\xi(0)$, $p_\eta(0)$ and $p_\varphi(0)$ is improved by Newton method. All numerical experiments are done in the area of small ν and λ near λ_c as suggested by Biezeno and Grammel [42]. Namely, the real compressed and twisted columns are loaded near Euler buckling load $\lambda_c = \pi^2$ and small value of twisting couple. In all the numerical calculations the value $\rho = 0.3$ is used. In Table 2. we present the minimum half volume $v_{\min}$, the minimal and maximal area of the column cross section, that are $a_{\min} = a(0)$ and $a_{\max} = a(1/2)$, obtained for a few values of ν and λ and for a few values of φ_p. The area of the uniform column of the same size as optimal a_s and the corresponding value of the twisting angle at trivial equilibrium configuration φ_s are also presented.

Table 2.

ν	λ	φ_p	$v_{\min}$	$a_{\max} = a(1/2)$	$a_{\min} = a(0)$	a_s	φ_s
0.05	9.868	0.073	0.436	1.147	0.266	0.871	0.043
		0.049	0.446	1.116	0.531	0.893	0.041
0.25	9.854	0.244	0.446	1.116	0.533	0.893	0.204
		0.179	0.480	1.042	0.866	0.960	0.177
1.0	9.620	0.878	0.454	1.098	0.641	0.907	0.790
		0.748	0.472	1.057	0.811	0.944	0.728

The optimal shapes that correspond to the numerical solutions of the column are presented in Figure 8. As in previous section instead of $a_{opt}(t)$ we present optimal curve obtained as $R_{opt}(t) = \sqrt{a_{opt}(t)}$, which according to Pearson formulation of the Lagrange problem, gives the compressed and twisted column of greatest efficiency.

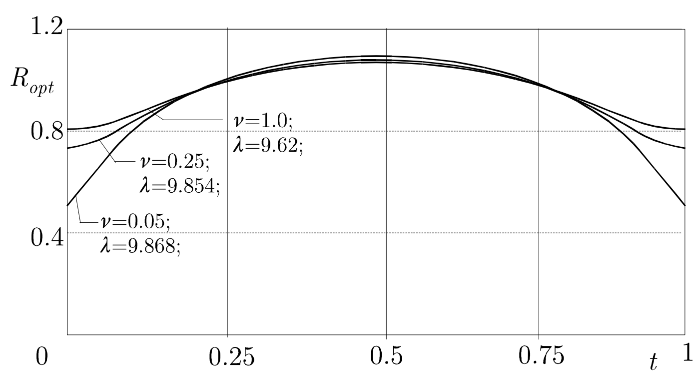

Figure 8. Optimal columns against spatial buckling.

For comparison we note that the columns presented in Figure 8. could still remain straight but the uniform columns made of the same amount of material, loaded with the same λ and ν are very far away in the postcritical region. We shall examine the postbuckling behavior of the column of uniform cross section $a_s = 0.907$, loaded with $\lambda = 9.62$ and $\nu = 1.0$. For these values the nonlinear two point boundary value problem (72), (73) is solved numerically. In Figure 9. we present the projection of the column axis C on the Oxy plane.

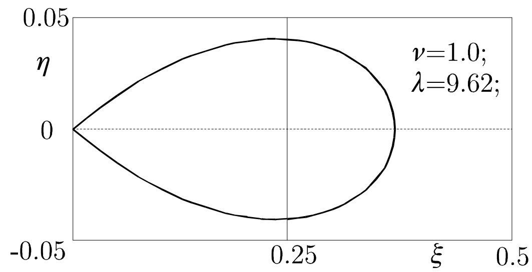

Figure 9. Postbuckling behavior of compressed and twisted uniform column.

Along the lines of the remarks of the previous section we note that instead of Kirchhoff's model of elastic rod we could use some other. Namely, we could analyze the effect of shear and compressibility in spatial buckling problem. In doing so either Haringx's, see Elyseyev [39] or Simo and Vu-Quoc [40], or Timoshenko type of decomposition, see Kingsbury [43], could be considered.

Finally, another note that could be related to the problem of Lagrange is connected with constrains on control variable. In application, optimal control problems where convexity assumption for the control domain is not needed are mathematically attractive as well as technically significant, see Nagahisa and Sakawa [44]. Roughly speaking, in the problem of Lagrange we could propose the strongest column to be made of only a few circular cross sections of different size, i.e., in our notation it means to impose constraint

$$a \in \{a_1, ..., a_l\},$$

for a few given real numbers $a_1, ..., a_l$. This could reduce the column weight as well as the expenses of column production. In practical terms the remarks above mean that the column problem allows some more formulations and still remains open.

4 Closure

We have presented two possible generalizations of the well known problem of determining the optimum shape of a column for which the buckling load is largest among all columns of given length and volume. First, the optimal shape of the column that can suffer not only flexure as in the classical elastica theory but also compression and shear is obtained. Our boundary value problem given by Eqs. (44) - (47) represents a considerable generalization of the column problem posed within the classical elastica theory. Consequently, the solution to our problem presented in Figure 5., generalizes Clausen's solution that corresponds to the classical theory. In attacking our problem we have employed several different, though equivalent formulations, of the governing equations in order to deal with different questions that could be correlated with the optimal shape of a column against buckling. Second, by use of Kirchhoff's kinetic analogue the distribution of material along the length of a compressed and twisted column hinged at either end is optimized so that the column is of minimum volume and will support a given load without spatial buckling. As before our boundary value problem given by Eqs. (77), (79), (80), (81) and (83) generalizes the classical column problem. Once again the solution to our problem, that is obtained by shooting method and presented in Figure 8., generalizes Clausen's solution. Necessary conditions for both problems we treated here are derived using the maximum principle of Pontryagin. The principal novelty of the present results is that both solutions, that follow from two possible generalizations of the classical Bernoulli-Euler bending theory, lead to the optimum column with non-zero cross sectional area at its ends. A few possible generalizations of the column problem are also discussed.

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Dear Prof. Spasic,

It is my pleasure to inform you that your paper entitled **Optimal Design of Elastic Columns For Maximum Buckling Load** has been accepted for publication as a book chapter in Volume 3 of Stability, Vibration and Control of Systems. My managing editor will contact you soon regarding the final camera-ready manuscript of your paper

Sincerely yours,

Ardéshir GURAN
Professor

CC: Prof Daniel Inman

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