OPTIMAL SHAPE OF AN ELASTIC ROD AGAINST BUCKLING UNDER COMPRESSION AND NON-CONSERVATIVE TORSION

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1 SUMMARY

An extension of the Lagrange problem: to find the curve which by its revolution about an axis in its plane determines the compressed and twisted rod of great efficiency is considered. The rod can suffer not only flexure but also compression and shear. Its ends are assumed to be attached to the supports by ideal spherical hinges and are free to rotate in any directions. The compressing force and twisting couple retain their initial directions during buckling what represents the non-conservative load. The nonlinear differential equilibrium equations describing spatial buckled state of the rod are derived, and analyzed by use of the adjacent equilibrium method. On the basis of the linearized equations the optimal curve is obtained numerically by use of the Pontryagin maximum principle. The principal novelty of the present results is the optimal column with non-zero cross sectional area at its ends, that generalize the Clausen solution in a natural way.

2 INTRODUCTION

The problem of determining that shape of compressed column which has the largest buckling load was posed by Lagrange in 1773. In 1851 Clausen solve it for columns of circular cross section pined at the end points. Although that result was mathematically correct the obtained optimal shape did have points where the cross section vanishes. In the work long unnoticed outside the Soviet Union, Nikolai was the first author who considered that anomaly of Clausen's solution, [1]. Roughly speaking, the optimal column at the ends does not recognize the difference between any applied finite load and, for example, its doubled value. It is obvious that this optimal shape has led to confusion and several attempts to resolve the anomaly. In order to avoid any finite load to induce infinite stresses in the column, Nikolai proposed minimal cross sectional area at the ends, determined so that given limiting stress will not be exceeded. Since then, many results of structural optimization could be related to the problem of Lagrange, see for example the references cited in [2], [3], [4], and [5].

In [4], two generalization of the problem of Lagrange were made. First, it was stated that Clausen's solution was correct in the sense of the Bernoulli-Euler elastica theory, and that if one wants to improve it to fit for real columns, the problem should be posed within a generalized plane elastica theory that takes into account both shear and compressibility effects. The solution obtained therein for the rod model developed in [6] shows that the optimal column at its ends depends on the load and material. As the second generalization of [4], twisting couples were added and the problem of Lagrange was generalized in a direction that leads to spatial buckling and the classical theory called Kirchhoff's kinetic analogue used. However, for that type of the rod model a stiff equation problem occurred as a part of the solution. The problem was avoided by proposing the twist angle at the middle of the column in advance, resulting in the optimal shape with non-zero cross sectional area at its ends.

In this paper we comment on the solution presented in [4]. Namely, considering the spatial optimal rod problem we find the arguments presented therein to be rather artificial. Thus we suggest a different strategy for the solution of the spatial generalization of the |Lagrange problem. Namely, to avoid the stiff equation problem in a natural way the different rod model will be adopted. As a new model we have in mind the spatially deformed rod that can suffer not only flexure as in classical elastica theory but also compression and shear, that is, we plan to use the results of [7], [8]. The nonlinear analysis, needed for establishing the model, relies on [9], where the generalization of the Grammel problem [10], considering the stability of a compressed and twisted rod was treated. The optimal control problem will be solved by use of the Pontryagin maximum principle. The principal novelty of the presented results will be the optimal column with non-zero cross sectional area at its ends obtained by some kind of a natural continuation method starting from the Clausen solution.

2 EQUILIBRIUM EQUATIONS AND THE STABILITY RESULT

Consider an elastic rod of length L, straight and prismatic in an undeformed state, hinged at both ends and loaded by the compressive force, say F, and two twisting couples of intensity W. The ends of the rod are assumed to be attached to the supports by ideal spherical hinges and are free to rotate in any directions. The compressing force and twisting couple retain their initial directions during buckling what represents the nonconservative load. The column is of a circular cross section of area $A = A(S)$, bending rigidity $EI = EI(S)$, and shear rigidity $GA = kGA(S)$, where S denotes the arc length of the rod axis measured from one end point and k represens shear correction factor that depends on the geometry of the cross section and on the material. Note that either $EI(S)$ or $A(S)$ determines the distribution of material along the length of the column. In the following we use A_o, GA_o, EA_o and EI_o to denote the cross-section area, shear, extensional and bending rigidity of the uniform (cylindrical) rod respectively to which the optimal one is to be compared.

Introducing the reference frame say $Oxyz$ at one end of the rod we describe the rod axis, i.e. the line (space curve) joining centroids of circular cross sections, by the functions $x(S)$, $y(S)$ and $z(S)$. At the centroid of any circle an orthogonal (local) coordinate system is constructed and then to each circle the Euler type angles $\psi(S)$, $\theta(S)$ and $\varphi(S)$ recording orientation of the local system to the reference frame are attached. The Euler angles describe any possible orientation in terms of a rotation ψ about the y axis, then rotation θ about the new x axis, and finally, a rotation about new z axis of φ. This type of Euler angles, usually called Krilov or "ship" angles is important because the small difference in orientation of the local and reference systems restricts the values of each of these angles. Analyzing the change of the local unit vectors along the line of centroids the elastic deformation of the rod can be described by the components of curvature and the twist. In order to obtain the differential equations describing the equilibrium of the rod we shall follow the standard procedure [11]. Namely, assuming that there are no static forces or moments distributed along the rod element, we have to write the condition of force equilibrium applied to the rod element as well as to sum moments about the arbitrary point of it. Then assuming linear elastic behavior the connection between the geometrical quantities and the component of the resultant force**,** and the resultant moment at arbitrary point S should be written. In doing so the same measure of deformation as proposed in [7], [8] (with no torsion-warping deformation), are taken. Noting that the functions $z(S)$ and $\varphi(S)$ could be omitted from the analysis, after some lengthy calculations, we obtain the nonlinear differential equations of the spatially deformed rod in the visual space as

$$\begin{aligned}
\frac{dx}{dS} &= \left(1+\left(\frac{F}{kGA}-\frac{F}{EA}\right)\cos\theta\cos\psi\right)\cos\theta\sin\psi, \\
\frac{dy}{dS} &= -\left(1+\left(\frac{F}{kGA}-\frac{F}{EA}\right)\cos\theta\cos\psi\right)\sin\theta, \\
\frac{d\psi}{dS} &= \frac{FL^2}{EI}\left(\left(y\sin\psi+\frac{W}{FL}\cos\psi\right)\tan\theta-x\right), \\
\frac{d\theta}{dS} &= \frac{FL^2}{EI}\left(y\cos\psi-\frac{W}{FL}\sin\psi\right),
\end{aligned} \tag{1}$$

The corresponding boundary conditions read

$$x(0)=0, \quad y(0)=0, \quad x(L)=0, \quad y(L)=0. \tag{2}$$

We note here that system (1), (2) is a special case of the system analyzed in [9] so the nonlinear analysis presented therein could be applied. By use of the method of adjacent equilibrium followed by the arguments presented in [12] it could be shown that the bifurcations of the nonlinear system (1) are determined by the bifurcation points of the corresponding linearized system. Taking into account the benefits of the adopted version of Euler's angles we write the linearized equilibrium equations of the rod

$$\begin{aligned}
\dot{x} &= \left(1+\frac{\beta-\mu}{a}\right)\psi, \quad \dot{y} = -\left(1+\frac{\beta-\mu}{a}\right)\theta, \\
\dot{\psi} &= \frac{\nu\theta-\lambda x}{a^2}, \quad \dot{\theta} = \frac{\lambda y-\nu\psi}{a^2},
\end{aligned} \tag{3}$$

where dot represents the derivative with respect to dimensionless Lagrange's coordinate, say $t = S/L \in [0,1]$; In (3) the dimensionless functions $x = x(t)$ and $y = y(t)$ determine the position of the column axis in space; $a(t) = A(t)/A_o$ denote the dimensionless area of the rod cross-section; $\lambda = FL^2/EI_o$; $\nu = WL/EI_o$; $\beta = F/kGA_o$, and $\mu = F/EA_o$. Note that $a^2 = EI/EI_o$.

The boundary conditions corresponding to (3) read

$$x(0)=0; \quad y(0)=0; \quad \psi(1/2)=0; \quad y(1/2)=0, \tag{4}$$

where as in [4] we have constrained ourselves to the first buckling mode in which the resultant moment achieve the maximum value at the middle of the column.

It could be shown that the critical load of the column of uniform cross-section is

$$\pi^2 = \lambda (1 + \beta - \mu) + \frac{\nu^2}{4}, \tag{5}$$

where we recognize the generalization of the well known result of Greenhill, [13]. We note that in [5] the different type of boundary conditions and consequently the different type of applied load, i.e., conservative torsion was analyzed.

4 OPTIMAL CONTROL PROBLEM

Now we may formulate the following optimal control problem: to find the distribution of the material along the length of a rod so that the rod is of minimum volume and will support a load for given ν and λ satisfying (5) without buckling, i.e.,

$$\min_a v;$$

subjected to (3), (4), where $v = \int_0^{1/2} a dt$, represents the dimensionless half volume of the column.

In solving the problem the Pontryagin maximum principle is used. Namely, we introduce the Lagrange multipliers $p_x, p_y, p_\psi, p_\theta$ to form Hamiltonian

$$H = a + p_x \left(1 + \frac{\beta - \mu}{a}\right) \psi - p_y \left(1 + \frac{\beta - \mu}{a}\right) \vartheta + p_\psi \left(\frac{\nu\theta - \lambda x}{a^2}\right) + p_\theta \left(\frac{\lambda y - \nu\psi}{a^2}\right).$$

According to that principle, necessary conditions for optimality, the Euler-Lagrange (costate) equations, and natural boundary conditions for the problem respectively read

$$\frac{\partial H}{\partial a} = 1 - \frac{(\beta - \mu) p_x \psi + (\mu - \beta) p_y \theta}{a^2} - 2\frac{p_\psi (\nu\theta - \lambda x) + p_\theta (\lambda y - \nu\psi)}{a^3} = 0, \tag{6}$$

$$\dot{p}_x = \frac{\lambda p_\psi}{a^2}, \quad \dot{p}_y = -\frac{\lambda p_\theta}{a^2},$$
$$\dot{p}_\psi = -p_x \left(1 + \frac{\beta - \mu}{a}\right) + \frac{\nu p_\theta}{a^2}, \quad \dot{p}_\theta = p_y \left(1 + \frac{\beta - \mu}{a}\right) - \frac{\nu p_\psi}{a^2}, \tag{7}$$

$$p_\psi (0) = 0, \quad p_\theta (0) = 0, \quad p_x (1/2) = 0, \quad p_\theta (1/2) = 0. \tag{8}$$

Note that at $S = 0$ (6) reduces to

$$a(0)^2 - (\beta - \mu) [p_x (0) \psi (0) - p_y (0) \vartheta (0)] = 0,$$

and the cross-section at $S = 0$ differs from zero, i.e.,

$$a(0) = \sqrt{(\beta - \mu) [p_x (0) \psi (0) - p_y (0) \vartheta (0)]},$$

that is, the optimal rod will recognize the material and the load at its ends. Thus, no artificial condition as in [4] is necessary and we may say that the optimal column problem will be solved in the natural way.

Finally, for $\lambda = 9.6663; \nu = 0.2; \beta = 0.03; \mu = 0.01$; we give a sketch of the numerical solution of the boundary value problem (3), (4), (7), (8), with a determined from (6), obtained by shooting method. Note that eq. (6) could be transformed into the polynomial equation which was solved exactly at each step of integration procedure. As recommended in [14], a real load of the compressed and twisted column is in the area of a relatively small value of the twisting couple and the compressing force not far from the Euler buckling load. This explains the chosen values of λ and ν.

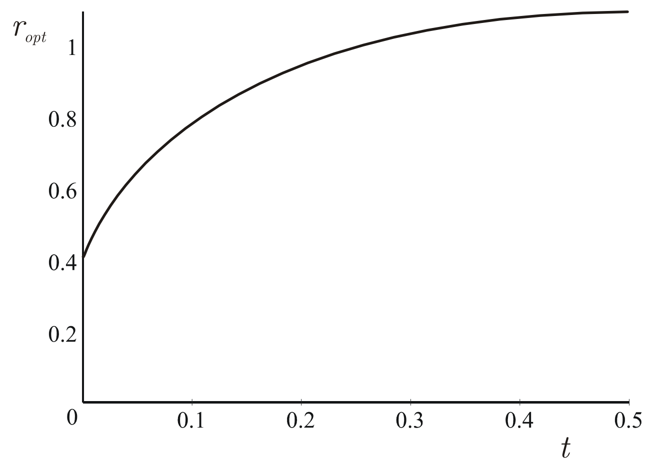

Figure 1: Optimal curve

In Figure 1 the optimal curve, obtained as $r_{opt}(t) = \sqrt{a_{opt}(t)}$, that determines the compressed and twisted column of greatest efficiency is shown. The obtained minimum half volume of the optimal column is $v_{\min} = 0.4384$. We conclude that the obtained solution is close to the Clausen solution, corresponding to $v = 0$, $\beta = 0$, $\mu = 0$, and represents its direct generalization.

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