FORCED VIBRATIONS WITH FRACTIONAL TYPE OF DISSIPATION

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1 SUMMARY
We consider the motion of a mass moving on a straight line under the action of a harmonic disturbing force. The mass is fixed to a viscoelastic body whose other end is anchored. It is assumed that the viscoelastic body behaves according to a generalized model that contains fractional derivatives of stress and strain. Thermodynamical restrictions on the coefficients of the model, that follow from the Clausius Duhem inequality, are taken into account. It is shown that the dynamics of the problem is governed by a single differential equation of real order. The obtained equation was solved by use of the method of Laplace transformation. The proposed model could be used for the study of forced vibrations of systems incorporating polymers, elastomers and other real materials.

2 INTRODUCTION
The study of forced vibrations is a classical problem. The interest to it increases if the materials included are taken to exhibit nonlinear behavior with, or without damping. As a part of it, the problem of eliminating undesirable oscillations and vibrations has emerged. Namely, the new tendency in civil engineering favors the design of slender structures as a consequence of the now available new high performance materials. The use of such materials requires thorough knowledge of their physical properties, especially the study of viscoelastic materials which provide necessary extra damping. Special feature of viscoelastic bodies is that there exists hysteresis like behavior in force displacement diagram that could be explained either by nonlinear models or by use of the standard linear viscoelastic model (the Zener model). According to Bagley [1] it seems that a generalized model of viscoelastic body that contains fractional derivatives of stress and strain (the generalized Zener model or the fractional standard viscoelastic body) is capable of describing the problem in a more accurate way while still remaining in linear theory. Recently, in the paper of Enelund and Lesieutre [2], an example of a forced oscillator including fractional damping elements was given. The solution was obtained by the Grünwald algorithm and the finite element method. The influence of the order of fractional derivative on the solution for one value of disturbing force frequency was considered.

In this work we intend to reexamine the problem. Namely, we are going to solve the problem by different methods. We plan to use the method of Laplace transformation with inversion performed by complex integration [3], [4]. As an alternative approaches we are going to show the numerical method described by Podlubny, [5] and Post's inversion formulae, [6], [7]. The alternative approaches are more convenient for engineers but are usually followed by problems concerning the convergence in large time domain and the short memory principle. Our analysis closes with the influence of four parameters, included in the viscoelastic material description, on the solution for different values of disturbing force frequencies. Namely, taking into account the restrictions on the parameters, that follow from Clausius Duhem inequality, the amplitude ratio (or magnification factor) will be analyzed. In doing so the resonance recognition problem will be tackled. In the analysis that follows the standard linear viscoelastic solid (the Zener model) will be treated as a special case of the fractional standard viscoelastic body.

3 FORMULATION OF THE PROBLEM

Consider a mass m moving on a straight line under the action of a harmonic disturbing force, say $F_0 \sin \Omega t$, where F_0 and Ω are positive constants and t is time. The mass is fixed to a viscoelastic body, which is assumed to be a rod of constant cross-sectional area A and of length $l.$ We assume that the other end of the rod is anchored and we use x to measure uniaxial, isothermal deformation of that rod. Let f be the force between the rod and the mass[1]. Applying the fundamental axiom of dynamics [8], we describe the considered motion by

$$mx^{(2)} = -f + F_0 \sin \Omega t, \qquad x(0) = 0, \quad x^{(1)}(0) = 0, \quad f(0) = 0, \tag{1}$$

where we used $(\cdot)^{(k)} = d^k(\cdot)/dt^k$ to denote the $k-$th derivative with respect to time, and where we assumed that the mass was at rest at initial time $t = 0$.

The relation between f=$f(t)$ and x=$x(t)$ (the constitutive relation of the deformable body) may be taken in different forms. In this paper we take it in the form that represents the generalized model of a viscoelastic body that contains fractional derivatives of stress and strain, i.e.

$$f + \tau_{f\alpha} \cdot f^{(\alpha)} = \frac{E_\alpha A}{l}\left(x + \tau_{x\alpha} \cdot x^{(\alpha)}\right), \tag{2}$$

where $0 < \alpha \leq 1$, E_α is the modulus of elasticity, $\tau_{f\alpha}$ and $\tau_{x\alpha}$ are the constants of dimension $[\text{time}]^\alpha$. In (2), for $0 < \alpha < 1$, we use $(\cdot)^{(\alpha)}$ to denote the $\alpha - th$ derivative of a function $(\cdot)$ taken in Riemann-Liouville form as $d^\alpha[g(t)]/dt^\alpha$=$g^{(\alpha)}$=$d[\Gamma^{-1}(1-\alpha)\int_0^t g(\xi)(t-\xi)^{-\alpha} d\xi]/dt$, where Γ denotes the Euler Gamma function. Note that in the special case when $\alpha = 1$ equation (2) represents the standard model of linear viscoelastic solid with τ_{f1} and τ_{x1} known as the relaxation times. Besides [2] the constitutive equation of the same type was used in [9], [10], [11] and [12] for example. Note that there exists fundamental restrictions on the coefficients of the model, that follow from the second law of thermodynamics $E_\alpha > 0,\ \tau_{f\alpha} > 0,\ \tau_{x\alpha} > \tau_{f\alpha}$,as proposed in the just mentioned papers.

Introducing the dimensionless coordinate, force, time and frequency of the excitation force, say $\bar{x} = xE_\alpha A(F_0 l)^{-1}$, $\bar{f} = fF_0^{-1}$, $\bar{t} = t\left[E_\alpha A/(ml)\right]^{1/2}$ and $\bar{\Omega} = \Omega\left[ml/(E_a A)\right]^{1/2}$ respectively, as well as the dimensionless constants $\bar{\tau}_{f\alpha} = \tau_{fa}\left[E_\alpha A/(ml)\right]^{\alpha/2}$ and $\bar{\tau}_{x\alpha} = \tau_{xa}\left[E_\alpha A/(ml)\right]^{\alpha/2}$, from (1) and (2) we get the system of equations describing the forced vibrations with fractional

[1] The force f used here is given as $f = A\sigma$ where A is the cross-sectional area and σ is the stress. We assume that the cross sectional area remains the same during the deformation.

type of dissipation

$$x^{(2)} = -f + \sin \Omega t, \qquad x(0) = 0,\ x^{(1)}(0) = 0,\ f(0) = 0,$$
$$f + \tau_{f\alpha} \cdot f^{(\alpha)} = x + \tau_{x\alpha} \cdot x^{(\alpha)}, \tag{3}$$

where the derivatives are taken with respect to dimensionless time. In the sequel the bars are suppressed over the dimensionless variables. Note that as a consequence of the second law of thermodynamics in (3) we have $\Delta\tau_\alpha = \tau_{x\alpha} - \tau_{f\alpha} > 0$ and there will be no damping if $\Delta\tau_\alpha = 0$, see [9]. Also, note that following the lines of the classical vibration theory, when $\tau_{f\alpha} = \tau_{x\alpha}$, we expect the resonance and the vibroisolation to be exhibited for $\Omega = 1$ and $\Omega \gg 1$ respectively.

The proposed fractional standard linear solid model could be effective in describing the behavior of some real materials (polymers, elastomers). Besides, it has an essential mathematical interest too. Thus one of the main results of this paper concerns the solution of (3). In the following section we are going to examine some of the methods which may be useful in many engineering applications, especially when materials involved exhibit hysteresis type of force-displacement behavior.

4 THE SOLUTION OF EQUATIONS

In order to compute the solution of (3) for the case $\alpha < 1$ we apply numerical method presented in [5], p. 223. First, we eliminate f, and then by use of basic properties of the Riemann-Liouville fractional differentiation, instead of (3) we obtain the following (single) differential equation of real order

$$\tau_{f\alpha} x^{(2+\alpha)} + x^{(2)} + x + \tau_{x\alpha} x^{(\alpha)} = \sin \Omega t + S_t(-\alpha, \Omega),$$
$$x(0) = 0, \quad x^{(1)}(0) = 0, \quad x^{(2)}(0) = 0, \tag{4}$$

where $S_t(-\alpha, \Omega) = \sum_{j=0}^{\infty} (-1)^j \Omega^{2j+1} t^{2j+1-\alpha} \Gamma^{-1}(\alpha + 2j + 2)$ stands for the $\alpha - th$ Riemann-Liouville derivative of $\sin \Omega t$, see [13], p. 355. Using the first order approximation of problem (4), according to [5], we derive the following algorithm for obtaining the numerical solution

$$x_m = x(t_m) = \frac{1}{1 + h^{-2} + \tau_{x\alpha} h^{-\alpha} + \tau_{f\alpha} h^{-2-\alpha}} \times$$
$$\left\{ \frac{2x_{m-1} - x_{m-2}}{h^2} - \frac{\tau_{x\alpha} \sum\limits_{j=1}^{m} \omega_{j,\alpha} x_{m-j}}{h^\alpha} - \frac{\tau_{f\alpha} \sum\limits_{j=1}^{m} \omega_{j,2+\alpha} x_{m-j}}{h^{2+\alpha}} \right. \tag{5}$$
$$\left. \sin(\Omega m h) + S_{mh}(-\alpha, \Omega), \right\}, \qquad m = 3, 4, ...$$

where h is time step ($t_m = mh$), and where the coefficients $\omega_{j,\kappa}$, $\kappa = \alpha, 2+\alpha$, are calculated by the recurrence relationships $\omega_{0,\kappa} = 1$ and $\omega_{j,\kappa} = (1 - (\kappa + j)/j)\, \omega_{j-1,\kappa},\ j = 1, 2, 3, ...$Note that homogeneous initial conditions $(4)_{2,3,4}$ correspond to $x_0 = x_1 = x_2 = 0$.

The described numerical method was experimentally verified on a number of test problems by comparing it (when it was possible) with analytical solutions, see [5]. In the case of equation (4), with $S_{mh}(-\alpha,\Omega)$ given as above, it seems that it will work provided the time mh does not leave the convergence domain of that series. Since we know that $S_t(-1,\Omega)$ coincide with $\Omega\cos(\Omega t)$, see [13], p.318, we may speculate that if $S_{mh}(-1,\Omega)$ does not coincide with $\Omega\cos(\Omega mh)$, for say $m>m_c$, then we are not to expect the series $S_{mh}(-\alpha,\Omega)$ to be convergent for $m>m_c$ and $\alpha<1$, and thus the algorithm (5) may fail for $t>t_c=m_c h$. This really does happen in practice. For example, the numerical examination shows that $\cos t$ and $S_t(-1,1)$, truncated after 80 terms, does not coincide for $t>30$. Another problem that could be encountered while processing (5) is the short memory problem. Namely, if we take h to be small enough for large values of m the number of the addends in the fractional-derivative approximation becomes enormously large, what causes some extra technical problems, see [5], p. 203.

Since we do not know the duration of the oscillator transient regime, by use of (5) we may, or may not, reach the steady state solution of the forced oscillator problem. This increases our interest in finding alternative algorithms. Thus, in the following we are going to apply the Laplace transform and Post's inversion formula.

Introducing $X{=}X(s){=}\mathcal{L}\{x(t)\}{=}\int_0^\infty e^{-st}x(t)\,dt$ and $F{=}F(s){=}\mathcal{L}\{f(t)\}{=}\int_0^\infty e^{-st}f(t)\,dt$, from $(3)_5$ we get

$$F=\frac{1+\tau_{x\alpha}s^\alpha}{1+\tau_{f\alpha}s^\alpha}X, \tag{6}$$

where we have used the standard expression for the Laplace transform of $z^{(\alpha)}$, given as $\mathcal{L}\{z^{(\alpha)}\}{=}s^\alpha Z\text{-}\left[\left(\int_0^t z(\xi)\,d\xi/(t-\xi)^\alpha\right)\right]_{t=0}$, where $\mathcal{L}\{z(t)\}{=}Z{=}Z(s)$ and the term in brackets vanishes if $\lim_{t\to 0^+}z(t)$ is bounded (see [14]). It could be shown that the inversion of (6) yields the following relation between $f(t)$ and $x(t)$

$$f(t)=\frac{\tau_{x\alpha}}{\tau_{f\alpha}}x(t)+\frac{1}{\tau_{f\alpha}}\left(1-\frac{\tau_{x\alpha}}{\tau_{f\alpha}}\right)\int_0^t e_{\alpha,\alpha}\left(t-\xi,\frac{1}{\tau_{f\alpha}}\right)x(\xi)\,d\xi, \tag{7}$$

where $e_{\alpha,\beta}(t;\lambda)$ stands for the generalized Mittag-Leffler function $e_{\alpha,\beta}(t;\lambda)\equiv E_{\alpha,\beta}(\text{-}\lambda t^\alpha)/t^{1-\beta}$ with $E_{\alpha,\beta}(t)=\sum_{n=0}^\infty t^n/\Gamma(\alpha n+\beta)$. This equation represents the force-displacement hysteresis behavior. It could be used if one wants to rewrite the one degree-of freedom forced oscillations system with fractional damping elements in the compact form of single integro-differential equation. Note that for $\alpha=1$ the inversion of (6) yields the following relation between the force and the coordinate $f(t)=\tau_{x1}x(t)/\tau_{f1}+(1-\tau_{x1}/\tau_{f1})\,\tau_{f1}^{-1}\int_0^t e^{-(t-\xi)/\tau_{f1}}x(\xi)\,d\xi$ which is generalized in (7) for $\alpha<1$ as expected. On the other hand transforming (3) and using (6) for $0<\alpha<1$ we get

$$X=\frac{\Omega}{(s^2+\Omega^2)}\frac{(1+\tau_{f\alpha}s^\alpha)}{(\tau_{f\alpha}s^{2+\alpha}+s^2+\tau_{x\alpha}s^\alpha+1)}. \tag{8}$$

Substituting (8) in (6) one can get F that is the Laplace transform of $f(t)$.

In the special case when $\alpha=1$, corresponding to the Zener model, the direct inversion of (8), easily performed by use of standard software packages, yields the solution $x(t)$. In the general case $\alpha < 1$ the standard software packages fail to proceed, but one could obtain both $x(t)$ and $f(t)$ by use of Post's inversion formula, see [6] p. 380, i.e.

$$x(t) = \lim_{n\to\infty} \frac{(-1)^n \left(\frac{n}{t}\right)^{n+1} X^{(n)}\left(\frac{n}{t}\right)}{n!}, \quad f(t) = \lim_{n\to\infty} \frac{(-1)^n \left(\frac{n}{t}\right)^{n+1} F^{(n)}\left(\frac{n}{t}\right)}{n!}.$$

Although Post's formula, discovered in 1930 [7], may be regarded as an analytical result, very useful for applications, difficulties essentially technical in nature prevented its usage in practical problems. However, nowadays the n-th derivatives of (8) needed for the right-hand-side of Post's formula could be easily calculated by use of standard software packages. In such a way we may obtain results useful for error estimations of numerical solutions. At the same time the Post result could serve as analytical approximation for $x(t)$ (and $f(t)$) provided the computer has enough memory and is fast enough to perform large amount of symbolic differentiation. As an illustration we note that in [12] the same constitutive model (2) was analyzed in the compliant contact-impact problem modelled by $x_{ci}^{(2)} = -f_{ci}$, $x_{ci}(0) = 0$, $x_{ci}^{(1)}(0) = 1$ and $f_{ci}(0) = 0$. We added index ci for the convenience. In [12] the inversion of the function $X_{ci} = (1 + \tau_{f\alpha} s^{\alpha}) \times (\tau_{f\alpha} s^{2+\alpha} + s^2 + \tau_{x\alpha} s^{\alpha} + 1)^{-1}$ was obtained by use of both numerical algorithm similar to (5) and the Post inversion formula. The agreement between the results was satisfactory even for relatively small values of n ($n = 40$ in case $\alpha < 1$ and $n = 70$ when $\alpha = 1$). Note that second multiplicand in (8) is the same as X_{ci} and that the other one stands for the Laplace transform of $\sin \Omega t$. Since X_{ci} was inverted in [12] the motion of the forced oscillator $x(t)$ could be obtained by the convolution $x(t) = \int_0^t x_{ci}(\psi) \sin(\Omega t - \psi)\, d\psi$, with x_{ci} obtained by Post's inversion formula applied to X_{ci}. This procedure avoids the problems connected with the series $S_t(-\alpha, \Omega)$ but, despite the simplicity of the Post inversion formula, we may speculate that both short memory and time consuming problems could occur before reaching the steady state regime of the forced oscillator. Thus we turn now to the most elegant solution.

In order to examine the motion of the forced oscillator the inversion of (8) by complex integration will be done. Following the standard procedure, [3] p. 259, first, we chose the contour with a cut along the negative real axis, say γ, as shown in Fig. 1 (the path $ABDEFGA$) . Then we analyze the number of poles of (8) inside γ. The poles $s_1 = j\Omega$ and $s_2 = -\Omega j$, where j stands for the imaginary unit, are obvious. In order to determine the other ones we apply Rouché's theorem, see [4], p. 287. Namely, rewriting the second multiplicand of (8), as $1/(p+q)$ where $p = s^2 + 1$ and $q = (\tau_{x\alpha} - \tau_{f\alpha}) s^{\alpha} / (1 + \tau_{f\alpha} s^{a})$, and noting that for $s = \rho e^{j\theta}$ the condition $|g| / |f| < 1$ is satisfied on γ we conclude that p and $p+q$ have the same numbers of zeros inside γ, (in our case 2).

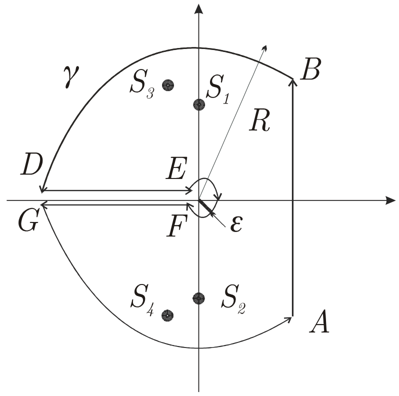

Figure 1: Contour of integration

In order to find two more poles of X we split the equation $\tau_{f\alpha}s^{2+\alpha}+s^2+\tau_{x\alpha}s^{\alpha}+1=0$ into the system

$$\begin{aligned} \tau_{f\alpha}\rho^{2+\alpha}\cos\left(2+\alpha\right)\theta+\rho^2\cos 2\theta+\tau_{x\alpha}\rho^{\alpha}\cos\alpha\theta+1=0, \\ \tau_{f\alpha}\rho^{2+\alpha}\sin\left(2+\alpha\right)\theta+\rho^2\sin 2\theta+\tau_{x\alpha}\rho^{\alpha}\sin\alpha\theta=0. \end{aligned} \tag{9}$$

Applying the Newton method we may find the solutions of (9) that correspond to principal branch, say $\bar{\rho}$ and $\bar{\theta}$ and the remaining poles of X, say $s_3=\bar{\rho}e^{\bar{\theta}j},\quad s_4=\bar{\rho}e^{-\bar{\theta}j}$. With this preparation done, we are ready to find $x\left(t\right)=\lim_{Y\to\infty}\left(2\pi j\right)^{-1}\int_{a-Yj}^{a+Yj}e^{st}X\left(s\right)ds$, $t>0$, i.e., as the integral along AB, where a is suitably chosen so all poles lie to the left of the line $s=a$. It remains to explore the residue theorem. According to it, the integral along the closed path γ is $2\pi j$ times the sum of the residues of $e^{st}X\left(s\right)$ at the singularities enclosed by γ. Rewriting $e^{st}X\left(s\right)$ as $F_1\left(s,t\right)/F_2\left(s\right)$ with

$$F_1\left(s,t\right)=\left(1+\tau_{f\alpha}s^{\alpha}\right)\Omega e^{st},\quad F_2\left(s\right)=\left(s^2+\Omega^2\right)\left(\tau_{f\alpha}s^{2+\alpha}+s^2+\tau_{x\alpha}s^{\alpha}+1\right), \tag{10}$$

the residue of $F_1\left(s,t\right)/F_2\left(s\right)$, at the point s_o, reads $F_1\left(s_o,t\right)/F_2'\left(s_o\right)$ where prime represents the derivative with respect to s, see [15] p. 161. Referring to Doetcsh once again, we conclude that the integrals along BD, GA and EF vanish (when $R\to\infty$ and $\varepsilon\to 0$). After calculating the sum of the integrals along DE and FG, we finally obtain the motion of the forced oscillator with fractional type of dissipation as

$$x\left(t\right)=\sum_{i=1}^{4}\frac{F_1\left(s_i,t\right)}{F_2'\left(s_i\right)}+\frac{\Omega\left(\tau_{x\alpha}-\tau_{fa}\right)\sin\alpha\pi}{\pi}\times I\left(t\right), \tag{11}$$

where

$$I\left(t\right)=\int_0^{\infty}\frac{\left(r^2+\Omega^2\right)^{-1}r^{\alpha}e^{-rt}dr}{\left(1+r^2\right)^2+\left(\tau_{f\alpha}r^{2+\alpha}+\tau_{x\alpha}r^{\alpha}\right)^2+2\left(1+r^2\right)\left(\tau_{f\alpha}r^{2+\alpha}+\tau_{x\alpha}r^{\alpha}\right)\cos\alpha\pi}. \tag{12}$$

Note that the residuals of X determine the value of $I\left(0\right)$. The value $I\left(t\right)$ could be easily calculated by standard procedures. Also note that $\lim_{t\to\infty}I\left(t\right)=0$.

6 RESULTS

In order to illustrate the above results we are going to present motions of the forced oscillator for numerical values of constants $0 < \alpha < 1$, $\tau_{f\alpha}$ and $\tau_{x\alpha}$ taken from the paper of Fenander where the railpad models were investigated, [9].

Namely, for $\alpha = 0.23$, $\tau_{f\alpha} = 0.004$, $\tau_{x\alpha} = 1.183$ and $\Omega = 1$, the solutions of (9) read $\bar{\rho} = 1.499$ and $\bar{\theta} = \pm 1.679$. Substituting these values into (11), (12) we obtain the amplitude of the steady state regime to be 0.818. Performing the same type of numerical experiments while increasing Ω we conclude that the system goes towards the vibroisolation area. For example for $\alpha = 0.23$, $\tau_{f\alpha} = 0.004$, $\tau_{x\alpha} = 1.183$ and $\Omega = 10$ the amplitude of x in the steady state is less then 0.01. When compared to the standard viscoelastic solid $\alpha = 1$ the system for $\alpha < 1$ exhibits smaller amplitudes what agrees with the results presented in [2]. Namely, noting that $\lim_{t\to\infty} \sum_{i=3}^{4} F_1(s_i,t)\,[F_2'(s_i)]^{-1} + I(t)\,\pi^{-1}\Omega\,(\tau_{x\alpha} - \tau_{f\alpha})\sin\alpha\pi \to 0$ we obtain the steady state solution in the form $x_s(t) = \sum_{i=1}^{2} F_1(s_i,t)\,[F_2'(s_i)]^{-1}$. The amplitude of it reads

$$A_s = \sqrt{\left(\frac{F_1(\Omega j,0)}{F_2'(\Omega j)} + \frac{F_1(-\Omega j,0)}{F_2'(-\Omega j)}\right)^2 + \left(\frac{F_1(\Omega j,0)}{F_2'(\Omega j)} - \frac{F_1(-\Omega j,0)}{F_2'(-\Omega j)}\right)^2}. \tag{13}$$

Calculating (13) for different values of Ω, α, $\tau_{x\alpha}$ and $\tau_{f\alpha}$ we may obtain the magnification factor for the oscillator with fractional type of dissipation. Since the dimension of the parameter space in the introduced model is 4 we omit here the usual graphical presentation of that factor. However with introduced $\Delta\tau_\alpha = \tau_{x\alpha} - \tau_{f\alpha} > 0$ we note that increasing the value $\Delta\tau_\alpha$ the value of A_s decreases. This fact could be very useful in engineering applications.

Finally, we may pose a question how will the system under consideration behave if the second law of thermodynamic is violated. Choosing $\Delta\tau_\alpha < 0$ and solving (9) yield $\bar{\theta} < \pi/2$ and thus $\lim\limits_{t\to\infty} \sum_{i=3}^{4} F_1(s_i,t)\,[F_2'(s_i)]^{-1} \to \infty$, what leads to a motion represented by time diverging function. The last comment deals with the possibility of obtaining time diverging functions. Once again we turn to equation (8). If the roots of (8) are imaginary and symmetrically displaced about the origin it is possible to have one of them coincide with Ω in which case we would have a second order pole at $s = \pm\Omega j$ and a diverging time function, [16], p.196. Namely, putting $\bar{\rho} = \Omega$ and $\bar{\theta} = \pi/2$ in (9), for $\alpha > 0$ one obtains

$$1 - \Omega^2 + \Omega^\alpha\left(\tau_{x\alpha} - \tau_{f\alpha}\Omega^2\right)\cos\frac{\alpha\pi}{2} = 0, \qquad \tau_{x\alpha} - \tau_{f\alpha}\Omega^2 = 0, \tag{14}$$

which can be satisfied only if $\Omega = 1$ and $\tau_{x\alpha} = \tau_{f\alpha}$. Indeed, if we put $\tau_{x\alpha} = \tau_{f\alpha}$ into (8) the straightforward inversion yields $x_{\tau_{x\alpha}=\tau_{f\alpha}}(t) = (\sin t - t\cos t)/2$. We close by noting that $\tau_{x\alpha} = \tau_{f\alpha}$ is never satisfied for thermodynamically well-behaved models and that for such models the resonance may not occur. This agrees with the classical linear theory with spring and dashpot as a model. Thus, as a consequence of the Clausius Duhem inequality, we claim that the time diverging functions are allowed only for the linearly elastic (Hookean) models. It is worth noting that "the experiments of 280 years have demonstrated amply for every solid substance examined with sufficient care, that the strain resulting from small applied stress is not a linear function thereof", see [17], p.155.

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